ACTIONS OF SYMBOLIC DYNAMICAL SYSTEMS ON C^*-ALGEBRAS II. SIMPLICITY OF C^*-SYMBOLIC CROSSED PRODUCTS AND SOME EXAMPLES

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Abstract. We have introduced a notion of C^*-symbolic dynamical system in [K. Matsumoto: Actions of symbolic dynamical systems on C^*-algebras, to appear in J. Reine Angew. Math.], that is a finite family of endomorphisms of a C^*-algebra with some conditions. The endomorphisms are indexed by symbols and yield both a subshift and a C^*-algebra of a Hilbert C^*-bimodule. The associated C^*-algebra with the C^*-symbolic dynamical system is regarded as a crossed product by the subshift. We will study a simplicity condition of the C^*-algebras of the C^*-symbolic dynamical systems. Some examples such as irrational rotation Cuntz-Krieger algebras will be studied.

1. Introduction

In [CK], J. Cuntz and W. Krieger have founded a close relationship between symbolic dynamics and C^*-algebras (cf.[C], [C2]). They constructed purely infinite simple C^*-algebras from irreducible topological Markov shifts. The C^*-algebras are called Cuntz-Krieger algebras.

In [Ma], the author introduced a notion of λ-graph system, whose matrix version is called symbolic matrix system. A λ-graph system is a generalization of finite labeled graph and presents a subshift. He constructed C^*-algebras from λ-graph systems [Ma2] as a generalization of the above Cuntz-Krieger algebras. A λ-graph system gives rise to a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of a unital commutative AF-C^*-algbera $\mathcal{A}_\mathfrak{L}$ with some conditions stated below. A C^*-*symbolic dynamical system*, introduced in [Ma6], is a generalization of λ-graph system. It is a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of a unital C^*-algebra $\mathcal{A}$ such that the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with $\mathcal{A}$. A finite labeled graph gives rise to a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that $\mathcal{A} = \mathbb{C}^N$ for some $N \in \mathbb{N}$. Conversely, if $\mathcal{A} = \mathbb{C}^N$, the C^*-symbolic dynamical system comes from a finite labeled graph. A λ-graph system $\mathfrak{L}$ gives rise to a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that $\mathcal{A}$ is $C(\Omega_\mathfrak{L})$ for some compact Hausdorff space $\Omega_\mathfrak{L}$

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with $\dim \Omega_{\mathfrak{L}} = 0$. Conversely, if $\mathcal{A}$ is $C(X)$ for a compact Hausdorff space X with $\dim X = 0$, the C^*-symbolic dynamical system comes from a λ-graph system.

A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a nontrivial subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$, that we will denote by Λ_ρ, over Σ and a Hilbert C^*-right $\mathcal{A}$-module $(\phi_\rho, \mathcal{H}^\rho_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$ that has an orthogonal finite basis $\{u_\alpha\}_{\alpha\in\Sigma}$ and a unital faithful diagonal left action $\phi_\rho : \mathcal{A} \to L(\mathcal{H}^\rho_{\mathcal{A}})$. It is called a Hilbert C^*-*symbolic bimodule over* $\mathcal{A}$, and written as $(\phi_\rho, \mathcal{H}^\rho_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$. By using general construction of C^*-algebras from Hilbert C^*-bimodules established by M. Pimsner [Pim] (cf. [Ka]), the author has introduced a C^*-algebra denoted by $\mathcal{A} \rtimes_\rho \Lambda$ from the Hilbert C^*-symbolic bimodule $(\phi_\rho, \mathcal{H}^\rho_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$, where Λ is the subshift Λ_ρ associated with $(\mathcal{A}, \rho, \Sigma)$. We call the algebra $\mathcal{A} \rtimes_\rho \Lambda$ *the* C^*-*symbolic crossed product of* $\mathcal{A}$ *by the subshift* Λ. If $\mathcal{A} = \mathbb{C}$, the subshift Λ is the full shift $\Sigma^{\mathbb{Z}}$, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the Cuntz algebra $\mathcal{O}_{|\Sigma|}$ of order $|\Sigma|$. If $\mathcal{A} = C(X)$ with $\dim X = 0$, there uniquely exists a λ-graph system $\mathfrak{L}$ up to equivalence such that the subshift Λ is presented by $\mathfrak{L}$ and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$. Conversely, for any subshift, that is presented by a λ-graph system $\mathfrak{L}$, there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that Λ_ρ is the subshift presented by $\mathfrak{L}$, the algebra $\mathcal{A}$ is $C(\Omega_{\mathfrak{L}})$ with $\dim \Omega_{\mathfrak{L}} = 0$, and the algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with $\mathfrak{L}$ ([Ma6]). If in particular, $\mathcal{A} = \mathbb{C}^n$, the subshift Λ is a sofic shift and $\mathcal{A} \rtimes_\rho \Lambda$ is a Cuntz-Krieger algebra.

In this paper, a condition called (I) on $(\mathcal{A}, \rho, \Sigma)$ is introduced as a generalization of condition (I) on the finite matrices of Cuntz-Krieger [CK] and on the λ-graph systems [Ma2]. Under the assumption that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I), the simplicity conditions of the algebra $\mathcal{A} \rtimes_\rho \Lambda$ is discussed in Section 3. We further study ideal structure of $\mathcal{A} \rtimes_\rho \Lambda$ from the view point of quotients of the C^*-symbolic dynamical systems in Section 4. Related discussions have been studied in Kajiwara-Pinzari-Watatani's paper [KPW] for the C^*-algebras of Hilbert C^*-bimodules (cf. [Kat], [MS], [Tom], etc.). They have studied simplicity condition and ideal structure of the C^*-algebras of Hilbert C^*-bimodules in terms of the language of the Hilbert C^*-bimodules. Our approach to study the algebras $\mathcal{A} \rtimes_\rho \Lambda$ is from the view point of C^*-symbolic dynamical systems, that is differnt from theirs. In Section 5, we will study pure infiniteness of the algebras $\mathcal{A} \rtimes_\rho \Lambda$. To obtain rich examples of the algebras $\mathcal{A} \rtimes_\rho \Lambda$, we will in Section 6 construct C^*-symbolic dynamical systems from a finite family of automorphisms $\alpha_i \in \mathrm{Aut}(\mathcal{B}), i = 1, \dots N$ on a unital C^*-algebra $\mathcal{B}$ and a C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ with $\Sigma = \{\alpha_1, \dots, \alpha_N\}$. The C^*-symbolic dynamical system is denoted by $(\mathcal{B} \otimes \mathcal{A}, \rho^{\Sigma\otimes}, \Sigma)$ that is the tensor product between two C^*-symbolic dynamical systems$(\mathcal{B}, \alpha, \Sigma)$ and $(\mathcal{A}, \rho, \Sigma)$. As examples of C^*-symbolic crossed products, continuous analogue of Cuntz-Krieger algebras called irrational rotation Cuntz-Krieger algebras denoted by $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ and irrational rotation Cuntz algebras denoted by $\mathcal{O}_{\theta_1,\dots,\theta_N}$ are studied in Sections 8 and 9. They belongs to the class of the C^*-algebras of continuous graphs by V. Deaconu ([De],[De2]). The fixed point algebras $\mathcal{F}_{\mathcal{G},\theta,\dots,\theta_N}$ of $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ under gauge actions are no longer AF-algebras. They are A$\mathbb{T}$-algebras. In particular, the fixed point algebras $\mathcal{F}_{\theta_1,\dots,\theta_N}$ of $\mathcal{O}_{\theta_1,\dots,\theta_N}$ under gauge actions are simple A$\mathbb{T}$-algebras of real rank zero with unique tracial state if and only if differnce of rotation angles $\theta_i - \theta_j$ is irrational for some $i, j = 1, \dots, N$ (Theorem 9.4).

Throughout this paper, we denote by $\mathbb{Z}_+$ and by $\mathbb{N}$ the set of nonnegative integers and the set of positive integers respectively. A homomorphism and an isomorphism between C^*-algebras mean a $*$-homomorphism and a $*$-isomorphism respectively.

An ideal of a C^*-algebra means a closed two sided $*$-ideal.

2. C^*-symbolic dynamical systems and their crossed products

Let $\mathcal{A}$ be a unital C^*-algebra. In what follows, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit $1_{\mathcal{A}}$ of $\mathcal{A}$. The unit $1_{\mathcal{A}}$ is denoted by 1 unless we specify. We denote by $\mathrm{End}(\mathcal{A})$ the set of all endomorphisms of $\mathcal{A}$. Let Σ be a finite set. A finite family of endomorphisms $\rho_\alpha \in \mathrm{End}(\mathcal{A}), \alpha \in \Sigma$ is said to be *essential* if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with $\mathcal{A}$. It is said to be *faithful* if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$. We note that $\{\rho_\alpha\}_{\alpha\in\Sigma}$ is faithful if and only if the homomorphism $\xi_\rho : a \in \mathcal{A} \longrightarrow [\rho_\alpha(a)]_{\alpha\in\Sigma} \in \oplus_{\alpha\in\Sigma}\mathcal{A}$ is injective.

Definition ([Ma6]). A C^*-*symbolic dynamical system* is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital C^*-algebra $\mathcal{A}$ and an essential and faithful finite family of endomorphisms ρ_α of $\mathcal{A}$ indexed by $\alpha \in \Sigma$.

Two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be isomorphic if there exist an isomorphism $\Phi : \mathcal{A} \to \mathcal{A}'$ and a bijection $\pi : \Sigma \to \Sigma'$ such that $\Phi\circ\rho_\alpha = \rho'_{\pi(\alpha)}\circ\Phi$ for all $\alpha \in \Sigma$. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$ over Σ such that a word $\alpha_1 \cdots \alpha_k$ of Σ is admissible for $\Lambda_{(\mathcal{A},\rho,\Sigma)}$ if and only if $(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$ ([Ma6;Proposition 2.1]). The subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$ will be denoted by Λ_ρ or simply by Λ in this paper.

Let $\mathcal{G} = (G, \lambda)$ be a left-resolving finite labeled graph with underlying finite directed graph $G = (V, E)$ and labeling map $\lambda : E \to \Sigma$ (see [LM; p.76]). Denote by $v_1, \dots, v_N$ the vertex set V. Assume that every vertex has both an incoming edge and an outgoing edge. Consider the N-dimensional commutative C^*-algebra $\mathcal{A}_{\mathcal{G}} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_N$ where each minimal projection E_i corresponds to the vertex v_i for $i = 1, \dots, N$. Define an $N \times N$-matrix for $\alpha \in \Sigma$ by

$$
(2.1) \quad A^{\mathcal{G}}(i, \alpha, j) = \begin{cases} 1 & \text{if there exists an edge } e \text{ from } v_i \text{ to } v_j \text{ with } \lambda(e) = \alpha, \\ 0 & \text{otherwise} \end{cases}
$$

for $i, j = 1, \dots, N$. We set $\rho^{\mathcal{G}}_\alpha(E_i) = \sum_{j=1}^N A^{\mathcal{G}}(i, \alpha, j)E_j$ for $i = 1, \dots, N, \alpha \in \Sigma$. Then $\rho^{\mathcal{G}}_\alpha, \alpha \in \Sigma$ define endomorphisms of $\mathcal{A}_{\mathcal{G}}$ such that $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ is a C^*-symbolic dynamical system such that the algebra $\mathcal{A}_{\mathcal{G}}$ is $\mathbb{C}^N$, and the subshift $\Lambda_{\rho^{\mathcal{G}}}$ is the sofic shift $\Lambda_{\mathcal{G}}$ presented by $\mathcal{G}$. Conversely, for a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, if $\mathcal{A}$ is $\mathbb{C}^N$, there exists a left-resolving labeled graph $\mathcal{G}$ such that $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ and $\Lambda_\rho = \Lambda_{\mathcal{G}}$ the sofic shift presented by $\mathcal{G}$ ([Ma6;Proposition 2.2]).

More generally let $\mathfrak{L}$ be a λ-graph system (V, E, λ, ι) over Σ (see [Ma]). Its vertex set V is $\cup_{l=0}^\infty V_l$. We equip V_l with discrete topology. We denote by $\Omega_{\mathfrak{L}}$ the compact Hausdorff space with $\dim \Omega_{\mathfrak{L}} = 0$ of the projective limit $V_0 \xleftarrow{\iota} V_1 \xleftarrow{\iota} V_2 \xleftarrow{\iota} \cdots$, as in [Ma2;Section 2]. The algebra $C(V_l)$ of all continuous functions on V_l, denoted by $\mathcal{A}_{\mathfrak{L},l}$, is the direct sum $\mathcal{A}_{\mathfrak{L},l} = \mathbb{C}E^l_1 \oplus \cdots \oplus \mathbb{C}E^l_{m(l)}$ where each minimal projection E^l_i corresponds to the vertex v^l_i for $i = 1, \dots, m(l)$. Let $\mathcal{A}_{\mathfrak{L}}$ be the commutative C^*-algebra $C(\Omega_{\mathfrak{L}}) = \lim_{l\to\infty}\{\iota_* : \mathcal{A}_{\mathfrak{L},l} \to \mathcal{A}_{\mathfrak{L},l+1}\}$. Let $A_{l,l+1}, l \in \mathbb{Z}_+$ be the matrices defined in [Ma2; Theorem A]. For a symbol $\alpha \in \Sigma$ we set

$$
(2.2) \qquad \rho^{\mathfrak{L}}_\alpha(E^l_i) = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j)E^{l+1}_j \qquad \text{for } i = 1, 2, \dots, m(l),
$$

so that $\rho^{\mathfrak{L}}_\alpha$ defines an endomorphism of $\mathcal{A}_{\mathfrak{L}}$. We have a C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ such that the C^*-algebra $\mathcal{A}_{\mathfrak{L}}$ is $C(\Omega_{\mathfrak{L}})$ with $\dim \Omega_{\mathfrak{L}} = 0$, and the subshift $\Lambda_{\rho^{\mathfrak{L}}}$ coincides with the subshift $\Lambda_{\mathfrak{L}}$ presented by $\mathfrak{L}$. Conversely, for a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, if the algebra $\mathcal{A}$ is $C(X)$ with $\dim X = 0$, there exists a λ-graph system $\mathfrak{L}$ over Σ such that the associated C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}, \rho, \Sigma)$ ([Ma6;Theorem 2.4]).

Let $\mathfrak{L}$ and $\mathfrak{L}'$ be predecessor-separated λ-graph systems over Σ and Σ' respectively. Then $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}_{\mathfrak{L}'}, \rho^{\mathfrak{L}'}, \Sigma')$ if and only if $\mathfrak{L}$ and $\mathfrak{L}'$ are equivalent. In this case, the presented subshifts $\Lambda_{\mathfrak{L}}$ and $\Lambda_{\mathfrak{L}'}$ are identified through a symbolic conjugacy. Hence the equivalence classes of the λ-graph systems are identified with the isomorphism classes of the C^*-symbolic dynamical systems of the commutative AF-algebras.

We say that *a subshift* Λ *acts on a* C^**-algebra* $\mathcal{A}$ if there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift Λ_ρ is Λ. For a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, we have a Hilbert C^*-bimodule $(\phi_\rho, \mathcal{H}^\rho_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$ called a Hilbert C^*-symbolic bimodule ([Ma6]). We then have a C^*-algebra by using the Pimsner's general construction of C^*-algebras from Hilbert C^*-bimodules [Pim] (cf. [Ka], see also [KPW], [KW], [Kat], [MS], [PWY], [Sch] etc.). We denote the C^*-algebra by $\mathcal{A} \rtimes_\rho \Lambda$, where Λ is the subshift Λ_ρ associated with $(\mathcal{A}, \rho, \Sigma)$. We call the algebra $\mathcal{A} \rtimes_\rho \Lambda$ *the* C^**-symbolic crossed product of* $\mathcal{A}$ *by the subshift* Λ.

Proposition 2.1([Ma6;Proposition 4.1]). *The* C^**-symbolic crossed product* $\mathcal{A} \rtimes_\rho \Lambda$ *is the universal* C^**-algebra* $C^*(x, S_\alpha; x \in \mathcal{A}, \alpha \in \Sigma)$ *generated by* $x \in \mathcal{A}$ *and partial isometries* $S_\alpha, \alpha \in \Sigma$ *subject to the following relations called* (ρ)*:*

$$\sum_{\beta\in\Sigma} S_\beta S_\beta^* = 1, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x), \qquad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x$$

for all $x \in \mathcal{A}$ *and* $\alpha \in \Sigma$*. Furthermore for* $\alpha_1, \dots, \alpha_k \in \Sigma$*, a word* $(\alpha_1, \dots, \alpha_k)$ *is admissible for the subshift* Λ *if and only if* $S_{\alpha_1} \cdots S_{\alpha_k} \neq 0$.

Assume that $\mathcal{A}$ is commutative. Then we know ([Ma6;Theorem 4.2])

(i) If $\mathcal{A} = \mathbb{C}$, the subshift Λ is the full shift $\Sigma^{\mathbb{Z}}$, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the Cuntz algebra $\mathcal{O}_{|\Sigma|}$ of order $|\Sigma|$.

(ii) If $\mathcal{A} = \mathbb{C}^N$ for some $N \in \mathbb{N}$, the subshift Λ is a sofic shift $\Lambda_{\mathcal{G}}$ presented by a left-resolving labeled graph $\mathcal{G}$, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is a Cuntz-Krieger algebra $\mathcal{O}_{\mathcal{G}}$ associated with the labeled graph. Conversely, for any sofic shift $\Lambda_{\mathcal{G}}$, that is presented by a left-resolving labeled graph $\mathcal{G}$, there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift is the sofic shift $\Lambda_{\mathcal{G}}$, the algebra $\mathcal{A}$ is $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the Cuntz-Krieger algebra $\mathcal{O}_{\mathcal{G}}$ associated with the labeled graph $\mathcal{G}$.

(iii) If $\mathcal{A} = C(X)$ with $\dim X = 0$, there uniquely exists a λ-graph system $\mathfrak{L}$ up to equivalence such that the subshift Λ is presented by $\mathfrak{L}$ and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$. Conversely, for any subshift $\Lambda_{\mathfrak{L}}$, that is presented by a left-resolving λ-graph system $\mathfrak{L}$, there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift is the subshift $\Lambda_{\mathfrak{L}}$, the algebra $\mathcal{A}$ is $C(\Omega_{\mathfrak{L}})$ with $\dim \Omega_{\mathfrak{L}} = 0$, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$.

3. Condition (I) for C^*-symbolic dynamical systems

The notion of condition (I) for finite square matrices with entries in $\{0,1\}$ has been introduced in [CK]. The condition gives rise to the uniqueness of the associated Cuntz-Krieger algebras under the canonical relations of the generating partial isometries. The condition has been generalized by many authors to corresponding conditions for generalizations of the Cuntz-Krieger algebras, for instance, infinite directed graphs ([KPRR]), infinite matrices with entries in $\{0,1\}$ ([EL]), Hilbert C^*-bimodules ([KPW]), etc. (see also [Re], [Ka2],[Tom2],etc.). The condition (I) for λ-graph systems has been also defined in [Ma2] to prove the uniqueness of the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ under the canonical relations of generators. In this section, we will introduce the notion of condition (I) for C^*-symbolic dynamical systems to prove the uniqueness of the C^*-algebras $\mathcal{A} \rtimes_\rho \Lambda$ under the relation (ρ). In [KPW], a condition called (I)-free has been introduced. The condition is similar condition to our condition (I). The discussions given in [KPW] is also similar ones to ours in this section. We will give complete descriptions in our discussions for the sake of completeness. Throughout this paper, for a subset F of a C^*-algebra $\mathcal{B}$, we denote by $C^*(F)$ the C^*-subalgebra of $\mathcal{B}$ generated by F.

In what follows, $(\mathcal{A}, \rho, \Sigma)$ denotes a C^*-symbolic dynamical system and Λ the associated subshift Λ_ρ. We denote by Λ^k the set of admissible words μ of Λ with length $|\mu| = k$. Put $\Lambda^* = \cup_{k=0}^{\infty} \Lambda^k$, where Λ^0 denotes the empty word. Let $S_\alpha, \alpha \in \Sigma$ be the partial isometries in $\mathcal{A} \rtimes_\rho \Lambda$ satisfying the relation (ρ) in Proposition 2.1. For $\mu = (\mu_1, \dots, \mu_k) \in \Lambda^k$, we put $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$ and $\rho_\mu = \rho_{\mu_k} \circ \cdots \circ \rho_{\mu_1}$. In the algebra $\mathcal{A} \rtimes_\rho \Lambda$, we set

$$\begin{aligned}
\mathcal{F}_\rho =& C^*(S_\mu x S_\nu^* : \mu, \nu \in \Lambda^*, |\mu| = |\nu|, x \in \mathcal{A}), \\
\mathcal{F}_\rho^k =& C^*(S_\mu x S_\nu^* : \mu, \nu \in \Lambda^k, x \in \mathcal{A}), \text{ for } k \in \mathbb{Z}_+ \quad \text{ and} \\
\mathcal{D}_\rho =& C^*(S_\mu x S_\mu^*, \mu \in \Lambda^*, x \in \mathcal{A}).
\end{aligned}$$

The identity $S_\mu x S_\nu^* = \sum_{\alpha \in \Sigma} S_{\mu\alpha} \rho_\alpha(x) S_{\nu\alpha}^*$ for $x \in \mathcal{A}$ and $\mu, \nu \in \Lambda^k$ holds so that the algebra $\mathcal{F}_\rho^k$ is embedded into the algebra $\mathcal{F}_\rho^{k+1}$ such that $\cup_{k \in \mathbb{Z}_+} \mathcal{F}_\rho^k$ is dense in $\mathcal{F}_\rho$. The gauge action $\hat{\rho}$ of the circle group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ on $\mathcal{A} \rtimes_\rho \Lambda$ is defined by $\hat{\rho}_z(x) = x$ for $x \in \mathcal{A}$ and $\hat{\rho}_z(S_\alpha) = z S_\alpha$ for $\alpha \in \Sigma$. The fixed point algebra of $\mathcal{A} \rtimes_\rho \Lambda$ under $\hat{\rho}$ is denoted by $(\mathcal{A} \rtimes_\rho \Lambda)^{\hat{\rho}}$. Let $\mathcal{E}_\rho : \mathcal{A} \rtimes_\rho \Lambda \longrightarrow (\mathcal{A} \rtimes_\rho \Lambda)^{\hat{\rho}}$ be the conditional expectaton defined by

$$\mathcal{E}_\rho(X) = \int_{z \in \mathbb{T}} \hat{\rho}_z(X) dz, \qquad X \in \mathcal{A} \rtimes_\rho \Lambda.$$

It is routine to check that $(\mathcal{A} \rtimes_\rho \Lambda)^{\hat{\rho}} = \mathcal{F}_\rho$.

Let $\mathcal{B}$ be a unital C^*-algebra. Suppose that there exist an injective homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving their units and a family $s_\alpha \in \mathcal{B}, \alpha \in \Sigma$ of partial isometries satisfying

$$\sum_{\beta \in \Sigma} s_\beta s_\beta^* = 1, \qquad s_\alpha^* \pi(x) s_\alpha = \pi(\rho_\alpha(x)), \qquad \pi(x) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. Put $\widetilde{\mathcal{A}} = \pi(\mathcal{A}) \subset$ and $\tilde{\rho}_\alpha(\pi(x)) = \pi(\rho_\alpha(x)), x \in \mathcal{A}$. We then have

Lemma 3.1. *The triple $(\widetilde{\mathcal{A}}, \tilde{\rho}, \Sigma)$ is a C^*-symbolic dynamical system such that the presented subshift $\Lambda_{\tilde{\rho}}$ is the same as the one $\Lambda(=\Lambda_\rho)$ presented by $(\mathcal{A}, \rho, \Sigma)$.*

Let $\mathcal{O}_{\pi,s}$ be the C^*-subalgebra of $\mathcal{B}$ generated by $\pi(x)$ and s_α for $x \in \mathcal{A}, \alpha \in \Sigma$. In the algebra $\mathcal{O}_{\pi,s}$, we set

$$
\begin{aligned}
\mathcal{F}_{\pi,s} =& C^*(s_\mu \pi(x) s_\nu^* : \mu, \nu \in \Lambda^*, |\mu| = |\nu|, x \in \mathcal{A}),\\
\mathcal{F}_{\pi,s}^k =& C^*(s_\mu \pi(x) s_\nu^* : \mu, \nu \in \Lambda^k, x \in \mathcal{A}) \text{ for } k \in \mathbb{Z}_+ \quad \text{ and}\\
\mathcal{D}_{\pi,s} =& C^*(s_\mu \pi(x) s_\mu^* : \mu \in \Lambda^*, x \in \mathcal{A}).
\end{aligned}
$$

By the universality of the algebra $\mathcal{A} \rtimes_\rho \Lambda$, the correspondence

$$
x \in \mathcal{A} \longrightarrow \pi(x) \in \widetilde{\mathcal{A}}, \qquad S_\alpha \longrightarrow s_\alpha, \quad \alpha \in \Sigma
$$

extends to a surjective homomorphism $\tilde{\pi} : \mathcal{A} \rtimes_\rho \Lambda \longrightarrow \mathcal{O}_{\pi,s}$.

Lemma 3.2. *The restriction of $\tilde{\pi}$ to the subalgebra $\mathcal{F}_\rho$ is an isomorphism from $\mathcal{F}_\rho$ to $\mathcal{F}_{\pi,s}$.*

Proof. It suffices to show that $\tilde{\pi}$ is injective on $\mathcal{F}_\rho^k$. Suppose that $\sum_{\mu,\nu\in\Lambda^k} s_\mu \pi(x_{\mu,\nu}) s_\nu^* = 0$ for $\sum_{\mu,\nu\in\Lambda^k} S_\mu x_{\mu,\nu} S_\nu^* \in \mathcal{F}_\rho$ with $x_{\mu,\nu} \in \mathcal{A}$. For $\xi, \eta \in \Lambda^k$, it follows that

$$
\pi(\rho_\xi(1) x_{\xi,\eta} \rho_\eta(1)) = s_\xi^* (\sum_{\mu,\nu\in\Lambda^k} s_\mu \pi(x_{\mu,\nu}) s_\nu^*) s_\eta = 0.
$$

As $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ is injective, one has $\rho_\xi(1) x_{\xi,\eta} \rho_\eta(1) = 0$ so that $S_\xi x_{\xi,\eta} S_\eta^* = 0$. This implies that $\sum_{\mu,\nu\in\Lambda^k} S_\mu x_{\mu,\nu} S_\nu^* = 0$. $\square$

Definition. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ satisfies *condition* (I) if there exists a unital increasing sequence

$$
\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}
$$

of C^*-subalgebras of $\mathcal{A}$ such that $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbb{Z}_+, \alpha \in \Sigma$ and the union $\cup_{l\in\mathbb{Z}_+} \mathcal{A}_l$ is dense in $\mathcal{A}$ and for $k, l \in \mathbb{N}$ with $k \le l$, there exists a projection $q_k^l \in \mathcal{D}_\rho \cap {\mathcal{A}_l}'(= \{x \in \mathcal{D}_\rho \mid xa = ax \text{ for } a \in \mathcal{A}_l\})$ such that

(i) $q_k^l a \ne 0$ for all nonzero $a \in \mathcal{A}_l$,

(ii) $q_k^l \phi_\rho^m(q_k^l) = 0$ for all $m = 1, 2, \dots, k$, where $\phi_\rho^m(X) = \sum_{\mu\in\Lambda^m} S_\mu X S_\mu^*$.

As the projection q_k^l belongs to the diagonal subalgebra $\mathcal{D}_\rho$ of $\mathcal{F}_\rho$, the condition (I) of $(\mathcal{A}, \rho, \Sigma)$ is intrinsically determined by $(\mathcal{A}, \rho, \Sigma)$ by virtue of Lemma 3.2.

If a λ-graph system $\mathfrak{L}$ over Σ satisfies condition (I), then $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ satisfies condition (I) (cf. [Ma2;lemma 4.1]).

We now assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I). We set for $k \le l$

$$
\mathcal{F}_{\rho,l}^k = C^*(S_\mu x S_\nu^* : \mu, \nu \in \Lambda^k, x \in \mathcal{A}_l).
$$

There exists an inclusion relation $\mathcal{F}_l^k \subset \mathcal{F}_{l'}^{k'}$ for $k \le k'$ and $l \le l'$. We put a projection $Q_k^l = \phi_\rho^k(q_k^l)$ in $\mathcal{D}_\rho$.

Lemma 3.3. *The map $X \in \mathcal{F}_{\rho,l}^k \longrightarrow Q_k^l X Q_k^l \in Q_k^l \mathcal{F}_{\rho,l}^k Q_k^l$ is a surjective isomorphism.*

Proof. As q_k^l commutes with $\mathcal{A}_l$, for $x \in \mathcal{A}_l$ and $\mu, \nu \in \Lambda^k$, we have

$$Q_k^l S_\mu x S_\nu^* = \sum_{\xi \in \Lambda^k} S_\xi q_k^l S_\xi^* S_\mu x S_\nu^* = S_\mu q_k^l S_\mu^* S_\mu x S_\nu^* = S_\mu x q_k^l S_\nu^*,$$

and similarly $S_\mu x S_\nu^* Q_k^l = S_\mu x q_k^l S_\nu^*$ so that Q_k^l commutes with $S_\mu x S_\nu^*$. Hence the map $X \in \mathcal{F}_{\rho,l}^k \longrightarrow Q_k^l X Q_k^l \in Q_k^l \mathcal{F}_{\rho,l}^k Q_k^l$ defines a surjective homomorphism. It remains to show that it is injective. Suppose that $Q_k^l(\sum_{\mu,\nu \in \Lambda^k} S_\mu x_{\mu,\nu} S_\nu^*) Q_k^l = 0$ for $X = \sum_{\mu,\nu \in \Lambda^k} S_\mu x_{\mu,\nu} S_\nu^*$ with $x_{\mu,\nu} \in \mathcal{A}_l$. For $\xi, \eta \in \Lambda^k$, one has

$$0 = S_\xi S_\xi^* Q_k^l (\sum_{\mu,\nu \in \Lambda^k} S_\mu x_{\mu,\nu} S_\nu^*) Q_k^l S_\eta S_\eta^* = Q_k^l S_\xi x_{\xi,\eta} S_\eta^*,$$

so that $0 = S_\xi^* Q_k^l S_\xi x_{\xi,\eta} S_\eta^* S_\eta = S_\xi^* S_\xi q_k^l \rho_\xi(1) x_{\xi,\eta} S_\eta^* S_\eta = q_k^l \rho_\xi(1) x_{\xi,\eta} \rho_\eta(1)$. Hence $\rho_\xi(1) x_{\xi,\eta} \rho_\eta(1) = 0$ by condition (I). Thus $S_\xi x_{\xi,\eta} S_\eta^* = 0$, so that $\sum_{\xi,\eta \in \Lambda^k} S_\xi x_{\xi,\eta} S_\eta^* = 0$. $\square$

Lemma 3.4. $Q_k^l S_\mu Q_k^l = 0$ *for* $\mu \in \Lambda^*$ *with* $|\mu| \le k \le l$.

Proof. By condition (I), we have $Q_k^l \phi_\rho^m(Q_k^l) = 0$ for $1 \le m \le k$. For $\mu \in \Lambda^*$ with $|\mu| \le k$, one has $\phi_\rho^{|\mu|}(Q_k^l) S_\mu = S_\mu Q_k^l S_\mu^* S_\mu = S_\mu Q_k^l$. Hence we have $0 = Q_k^l \phi_\rho^{|\mu|}(Q_k^l) S_\mu = Q_k^l S_\mu Q_k^l$. $\square$

As a result, we have

Lemma 3.5. *The projections Q_k^l in $\mathcal{D}_\rho$ satisfy the following conditions:*

(a) $Q_k^l F - F Q_k^l$ *converges to* 0 *as* $k, l \to \infty$ *for* $F \in \mathcal{F}_\rho$.
(b) $\|Q_k^l F\|$ *converges to* $\|F\|$ *as* $k, l \to \infty$ *for* $F \in \mathcal{F}_\rho$.
(c) $Q_k^l S_\mu Q_k^l = 0$ *for* $\mu \in \Lambda^*$ *with* $|\mu| \le k \le l$.

We note that $Q_k^l S_\mu Q_k^l = 0$ if and only if $Q_k^l S_\mu Q_k^l S_\mu^* = 0$. Since $Q_k^l S_\mu Q_k^l S_\mu^*$ belongs to the algebra $\mathcal{F}_\rho$, the condition $Q_k^l S_\mu Q_k^l = 0$ is determined in the algebraic structure of $\mathcal{F}_\rho$. As the restriction of $\tilde{\pi} : \mathcal{A} \rtimes_\rho \Lambda \longrightarrow \mathcal{O}_{\pi,s}$ to $\mathcal{F}_\rho$ yields an isomorphism onto $\mathcal{F}_{\pi,s}$, by putting $\widetilde{Q}_k^l = \tilde{\pi}(Q_k^l)$ we have

Lemma 3.6. *The projections $\widetilde{Q}_k^l$ in $\mathcal{D}_{\pi,s}$ satisfy the following conditions:*

(a') $\widetilde{Q}_k^l F - F \widetilde{Q}_k^l$ *converges to* 0 *as* $k, l \to \infty$ *for* $F \in \mathcal{F}_{\pi,s}$.
(b') $\|\widetilde{Q}_k^l F\|$ *converges to* $\|F\|$ *as* $k, l \to \infty$ *for* $F \in \mathcal{F}_{\pi,s}$.
(c') $\widetilde{Q}_k^l s_\mu \widetilde{Q}_k^l = 0$ *for* $\mu \in \Lambda^*$ *with* $|\mu| \le k \le l$.

Proposition 3.7. *There exists a conditional expectation $\mathcal{E}_{\pi,s} : \mathcal{O}_{\pi,s} \longrightarrow \mathcal{F}_{\pi,s}$ such that $\mathcal{E}_{\pi,s} \circ \tilde{\pi} = \tilde{\pi} \circ \mathcal{E}_\rho$.*

Proof. Let $\mathcal{P}_{\pi,s}$ be the $*$-subalgebra of $\mathcal{O}_{\pi,s}$ generated algebraically by $\pi(x), s_\alpha$ for $x \in \mathcal{A}, \alpha \in \Sigma$. Then any $X \in \mathcal{P}_{\pi,s}$ can be written as a finite sum

$$X = \sum_{|\nu| \ge 1} X_{-\nu} s_\nu^* + X_0 + \sum_{|\mu| \ge 1} s_\mu X_\mu \quad \text{for some } X_{-\nu}, X_0, X_\mu \in \mathcal{F}_{\pi,s}.$$

Thanks to the previous lemma and a usual argument of [CK], the element $X_0 \in \mathcal{F}_{\pi,s}$ is unique for $X \in \mathcal{P}_{\pi,s}$ and the inequality $\|X_0\| \le \|X\|$ holds. The map $X \in \mathcal{P}_{\pi,s} \to X_0 \in \mathcal{F}_{\pi,s}$ can be extended to the desired expectation $\mathcal{E}_{\pi,s} : \mathcal{O}_{\pi,s} \longrightarrow \mathcal{F}_{\pi,s}$. $\square$

Therefore we have

Theorem 3.8. *Assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I). The homomorphism $\tilde{\pi} : \mathcal{A} \rtimes_\rho \Lambda \longrightarrow \mathcal{O}_{\pi,s}$ defined by*

$$\tilde{\pi}(x) = \pi(x), \quad x \in \mathcal{A}, \qquad \tilde{\pi}(S_\alpha) = s_\alpha, \quad \alpha \in \Sigma.$$

becomes a surjective isomorphism, and hence the C^*-algebras $\mathcal{A} \rtimes_\rho \Lambda$ and $\mathcal{O}_{\pi,s}$ are canonically isomorphic through $\tilde{\pi}$.

Proof. The map $\tilde{\pi} : \mathcal{F}_\rho \to \mathcal{F}_{\pi,s}$ is isomorphic and satisfies $\mathcal{E}_{\pi,s} \circ \tilde{\pi} = \tilde{\pi} \circ \mathcal{E}_\rho$. Since $\mathcal{E}_\rho : \mathcal{A} \rtimes_\rho \Lambda \longrightarrow \mathcal{F}_\rho$ is faithful, a routine argument shows that the homomorphism $\tilde{\pi} : \mathcal{A} \rtimes_\rho \Lambda \longrightarrow \mathcal{O}_{\pi,s}$ is actually an isomorphism. $\square$

Hence the following uniqueness of the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ holds.

Theorem 3.9. *Assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I). The C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the unique C^*-algebra subject to the relation (ρ). This means that if there exist a unital C^*-algebra $\mathcal{B}$ and an injective homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ and a family $s_\alpha \in \mathcal{B}, \alpha \in \Sigma$ of nonzero partial isometries satisfying the folloing relations:*

$$\sum_{\beta\in\Sigma} s_\beta s_\beta^* = 1, \qquad s_\alpha^* \pi(x) s_\alpha = \pi(\rho_\alpha(x)), \qquad \pi(x) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$, then the correspondence

$$x \in \mathcal{A} \longrightarrow \pi(x) \in \mathcal{B}, \qquad S_\alpha \longrightarrow s_\alpha \in \mathcal{B}$$

extends to an isomorphism $\tilde{\pi}$ from $\mathcal{A} \rtimes_\rho \Lambda$ onto the C^*-subalgebra $\mathcal{O}_{\pi,s}$ of $\mathcal{B}$ generated by $\pi(x), x \in \mathcal{A}$ and $s_\alpha, \alpha \in \Sigma$.

As a corollary we have

Corollary 3.10. *Assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I). For any nontrivial ideal $\mathcal{I}$ of $\mathcal{A} \rtimes_\rho \Lambda$, one has $\mathcal{I} \cap \mathcal{A} \neq \{0\}$.*

Proof. Suppose that $\mathcal{I} \cap \mathcal{A} = \{0\}$. Hence $S_\alpha \notin \mathcal{I}$ for all $\alpha \in \Sigma$. By Theorem 3.9, the quotient map $q : \mathcal{A} \rtimes_\rho \Lambda \longrightarrow \mathcal{A} \rtimes_\rho \Lambda / \mathcal{I}$ must be injective so that $\mathcal{I}$ is trivial. $\square$

Let $\lambda_\rho : \mathcal{A} \to \mathcal{A}$ be the completely positive map on $\mathcal{A}$ defined by $\lambda_\rho(x) = \sum_{\alpha\in\Sigma} \rho_\alpha(x)$ for $x \in \mathcal{A}$.

Definition. $(\mathcal{A}, \rho, \Sigma)$ is said to be *irreduible* if there exists no nontrivial ideal of $\mathcal{A}$ invariant under λ_ρ.

Therefore we have

Corollary 3.11. *Assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I). If $(\mathcal{A}, \rho, \Sigma)$ is irreducible, the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is simple.*

4. Quotients of C^*-symbolic dynamical systems

In this section, we will study ideal structure of the C^*-symbolic crossed products $\mathcal{A} \rtimes_\rho \Lambda$, related to quotients of C^*-symbolic dynamical systems. The ideal structure of C^*-algebras of Hilbert C^*-bimodules has been studied in Kajiwara, Pinzari and Watatani's paper [KPW] (cf. [Kat3]). Their paper is written in the language of Hilbert C^*-bimodules. In this section we will directly study ideal structure of the C^*-symbolic crossed products $\mathcal{A} \rtimes_\rho \Lambda$ by using the language of C^*-symbolic dynamical systems. We fix a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$.

An ideal J of $\mathcal{A}$ is said to be *ρ-invariant* if $\rho_\alpha(J) \subset J$ for all $\alpha \in \Sigma$. It is said to be *saturated* if $\rho_\alpha(x) \in J$ for all $\alpha \in \Sigma$ implies $x \in J$.

Lemma 4.1. *Let J be an ideal of $\mathcal{A}$.*

(i) *J is ρ-invariant if and only if $\lambda_\rho(J) \subset J$.*
(ii) *J is saturated if and only if $\lambda_\rho(a) \in J$ for $0 \le a \in \mathcal{A}$ implies $a \in J$.*

Proof. (i) Suppose that J satisfies $\lambda_\rho(J) \subset J$. For $x \in J$ one has $\lambda_\rho(x^*x) \ge \rho_\alpha(x^*x) = \rho_\alpha(x)^*\rho_\alpha(x)$ so that $\rho_\alpha(x)^*\rho_\alpha(x) \in J$ because ideal is hereditary. Hence $\rho_\alpha(x)$ belongs to J. The only if part is clear.

(ii) Suppose that J is saturated and $\lambda_\rho(a) \in J$ for $0 \le a \in \mathcal{A}$. Since $\lambda_\rho(a) \ge \rho_\alpha(a)$ and J is hereditary, one has $a \in J$. Conversely suppose that $x \in \mathcal{A}$ satisfies $\rho_\alpha(x) \in J$ for all $\alpha \in \Sigma$. As $\lambda_\rho(x^*x) = \sum_{\alpha\in\Sigma} \rho_\alpha(x)^*\rho_\alpha(x)$, $\lambda_\rho(x^*x)$ belongs to J. Hence the condition of the if part implies that $x^*x \in J$ so that $x \in J$. $\square$

Let J be a ρ-invariant saturated ideal of $\mathcal{A}$. We denote by $\mathcal{I}_J$ the ideal of $\mathcal{A} \rtimes_\rho \Lambda$ generated by J.

Lemma 4.2. *The ideal $\mathcal{I}_J$ is the closure of linear combinations of elements of the form $S_\mu c_{\mu,\nu} S_\nu^*$ for $c_{\mu,\nu} \in J$.*

Proof. Elements x and y of $\mathcal{A} \rtimes_\rho \Lambda$ are approximated by finite sums of elements of the form $S_\mu a_{\mu,\nu} S_\nu^*$ and $S_\xi b_{\xi,\eta} S_\eta^*$ for $a_{\mu,\nu}, b_{\xi,\eta} \in \mathcal{A}$ respectively. Hence xcy is approximated by elemnts of the form

$$\sum_{\mu,\nu} S_\mu a_{\mu,\nu} S_\nu^* \cdot c \cdot \sum_{\xi,\eta} S_\xi b_{\xi,\eta} S_\eta^* = \sum_{\mu,\nu,\xi,\eta} S_\mu a_{\mu,\nu} S_\nu^* c S_\xi b_{\xi,\eta} S_\eta^*.$$

In case of $|\nu| \ge |\xi|$, one has $\nu = \bar\nu\nu'$ with $|\bar\nu| = |\xi|$ so that

$$S_\nu^* c S_\xi = \begin{cases} S_{\nu'}^* \rho_{\bar\nu}(c) & \text{if } \bar\nu = \xi, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $S_\mu a_{\mu,\nu} S_\nu^* c S_\xi b_{\xi,\eta} S_\eta^*$ is $S_\mu a_{\mu,\nu} \rho_{\nu'}(\rho_{\bar\nu}(c) b_{\xi,\eta}) S_{\eta\nu'}^*$ or zero. Since J is ρ-invariant, it is of the form $S_\mu c_{\mu,\nu} S_{\eta\nu'}^*$ for some $c_{\mu,\nu} \in J$ or zero. The argument in case of $|\nu| \le |\xi|$ is similar. Since the ideal $\mathcal{I}_J$ is the closure of elements of the form $\sum_{i=1}^n x_i c_i y_i$ for $x_i, y_i \in \mathcal{A} \rtimes_\rho \Lambda$ and $c_i \in J$, the assertion is proved. $\square$

We set

$$\mathcal{D}_J = C^*(S_\mu c_\mu S_\mu^* : \mu \in \Lambda^*, c_\mu \in J),$$
$$\mathcal{D}_J^k = C^*(S_\mu c_\mu S_\mu^* : \mu \in \Lambda^*, |\mu| \le k, c_\mu \in J) \quad \text{for } k \in \mathbb{Z}_+.$$

Lemma 4.3.

(i) $\mathcal{D}_J = \mathcal{I}_J \cap \mathcal{D}_\rho$ *and hence* $\mathcal{D}_J \cap \mathcal{A} = \mathcal{I}_J \cap \mathcal{A}$.

(ii) $\mathcal{D}_J^k \cap \mathcal{A} = J$ *for* $k \in \mathbb{Z}_+$.

Proof. (i) Since the elements of the finite sum $\sum_\mu S_\mu c_\mu S_\mu^*$ for $c_\mu \in J$ are contained in $\mathcal{I}_J \cap \mathcal{D}_\rho$, the inclusion relation $\mathcal{D}_J \subset \mathcal{I}_J \cap \mathcal{D}_\rho$ is clear. Let $\mathcal{I}_J^{\rm alg}$ and $\mathcal{D}_J^{\rm alg}$ be the algebraic linear spans of $S_\mu c_{\mu,\nu} S_\nu^*$ for $c_{\mu,\nu} \in J$ and $S_\mu c_\mu S_\mu^*$ for $c_\mu \in J$ respectively. For any $x \in \mathcal{I}_J \cap \mathcal{D}_\rho$ take $x_n \in \mathcal{I}_J^{\rm alg}$ such that $\|x_n - x\| \to 0$. Let $\mathcal{E}_\rho : \mathcal{A} \rtimes_\rho \Lambda \to \mathcal{F}_\rho$ be the conditional expectation defined previously, and $\mathcal{E}_D : \mathcal{F}_\rho \to \mathcal{D}_\rho$ the conditional expectation defined by taking diagonal elements. The composition $\mathcal{E}_{\mathcal{D}_\rho} = \mathcal{E}_D \circ \mathcal{E}_\rho$ is the conditional expectation from $\mathcal{A} \rtimes_\rho \Lambda$ to $\mathcal{D}_\rho$ that satisfies $\mathcal{E}_{\mathcal{D}_\rho}(\mathcal{I}_J^{\rm alg}) = \mathcal{D}_J^{\rm alg}$. Since $\mathcal{E}_{\mathcal{D}_\rho}(x) = x$ and the inequality $\|x - \mathcal{E}_{\mathcal{D}_\rho}(x_n)\| \le \|x - x_n\|$ holds, x belongs to the closure $\mathcal{D}_J$ of $\mathcal{D}_J^{\rm alg}$. Hence we have $\mathcal{I}_J \cap \mathcal{D}_\rho \subset \mathcal{D}_J$ so that $\mathcal{D}_J = \mathcal{I}_J \cap \mathcal{D}_\rho$. As $\mathcal{A}$ is a subalgebra of $\mathcal{D}_\rho$, the equality $\mathcal{D}_J \cap \mathcal{A} = \mathcal{I}_J \cap \mathcal{A}$ holds.

(ii) An element $x \in \mathcal{D}_J^k$ is of the form $\sum_{|\mu| \le k} S_\mu c_\mu S_\mu^*$ for $c_\mu \in J$. As $S_\nu c_\nu S_\nu^* = \sum_{\alpha \in \Sigma} S_{\nu\alpha} \rho_\alpha(c_\nu) S_{\nu\alpha}^*$ and J is ρ-invariant, x can be written as $x = \sum_{|\nu|=k} S_\nu c_\nu S_\nu^*$ for $c_\nu \in J$, and the element $\lambda_\rho^k(x) = \sum_{|\nu|=k} \rho_\nu(1) c_\nu \rho_\nu(1)$ belongs to J. Further suppose that x is an element of $\mathcal{A}$. Since J is saturated, by Lemma 4.1, one has $x \in J$. Hence the inclusion relation $\mathcal{A} \cap \mathcal{D}_J^k \subset J$ holds. The converse inclusion relation is clear so that $\mathcal{A} \cap \mathcal{D}_J^k = J$. □

Lemm 4.4. $\mathcal{A} \cap \mathcal{D}_J = J$.

Proof. Since the inclusion relation $\mathcal{A} \cap \mathcal{D}_J \supset J$ is clear, there exists a natural surjective homomorphism from $\mathcal{A}/J$ onto $\mathcal{A}/\mathcal{A} \cap \mathcal{D}_J$. For an element a of a C^*-algebra $\mathcal{B}$, we denote by $\|[a]_{\mathcal{B}/I}\|$ the norm of the quotient image $[a]_{\mathcal{B}/I}$ of a in the quotient $\mathcal{B}/I$ of $\mathcal{B}$ by an ideal I. As the inclusion $\mathcal{A} \hookrightarrow \mathcal{D}_\rho$ induces the inclusions both $\mathcal{A}/\mathcal{A} \cap \mathcal{D}_J \hookrightarrow \mathcal{D}_\rho/\mathcal{D}_J$ and $\mathcal{A}/\mathcal{A} \cap \mathcal{D}_J^k \hookrightarrow \mathcal{D}_\rho/\mathcal{D}_J^k$, one has for $a \in \mathcal{A}$

$$\|[a]_{\mathcal{A}/\mathcal{A}\cap\mathcal{D}_J}\| = \|[a]_{\mathcal{D}_\rho/\mathcal{D}_J}\|, \qquad \|[a]_{\mathcal{A}/\mathcal{A}\cap\mathcal{D}_J^k}\| = \|[a]_{\mathcal{D}_\rho/\mathcal{D}_J^k}\|.$$

Note that $\mathcal{D}_J$ is the inductive limit of $\mathcal{D}_J^k, k = 0, 1, \dots$. It then follows that

$$\|[a]_{\mathcal{D}_\rho/\mathcal{D}_J}\| = \operatorname{dist}(a, \mathcal{D}_J) = \lim_{k\to\infty} \operatorname{dist}(a, \mathcal{D}_J^k) = \lim_{k\to\infty} \|[a]_{\mathcal{D}_\rho/\mathcal{D}_J^k}\| = \lim_{k\to\infty} \|[a]_{\mathcal{A}/\mathcal{A}\cap\mathcal{D}_J^k}\|$$

and hence $\|[a]_{\mathcal{A}/\mathcal{A}\cap\mathcal{D}_J}\| = \|[a]_{\mathcal{A}/J}\|$ by Lemma 4.3 (ii). Thus the quotient map $\mathcal{A}/J \to \mathcal{A}/\mathcal{A} \cap \mathcal{D}_J$ is isometric so that $\mathcal{A} \cap \mathcal{D}_J = J$. □

By Lemma 4.3 and Lemma 4.4, one has

Proposition 4.5. $\mathcal{I}_J \cap \mathcal{A} = J$.

We will now consider quotient C^*-symbolic dynamical systems. Let J be a ρ-invariant saturated ideal of $\mathcal{A}$. We set $\Sigma_J = \{\alpha \in \Sigma \mid \rho_\alpha(1) \notin J\}$. We denote by $[x]$ the class of $x \in \mathcal{A}$ in the quotient $\mathcal{A}/J$. Put

$$\rho_\alpha^J([x]) = [\rho_\alpha(x)] \qquad \text{for } [x] \in \mathcal{A}/J, \quad \alpha \in \Sigma_J.$$

As J is ρ-invariant and saturated, ρ_α^J is well-defined and the family $\{\rho_\alpha^J\}_{\alpha\in\Sigma_J}$ is a faithful and essential endomorphisms of $\mathcal{A}/J$. We call the C^*-symbolic dynamical

system $(\mathcal{A}/J, \rho^J, \Sigma_J)$ the quotient of $(\mathcal{A}, \rho, \Sigma)$ by the ideal J. We denote by Λ_J the associated subshift for the quotient $(\mathcal{A}/J, \rho^J, \Sigma_J)$.

Definition. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to satisfy *condition* (II) if for any proper ρ-invariant saturated ideal J of $\mathcal{A}$, the quotient C^*-symbolic dynamical system $(\mathcal{A}/J, \rho^J, \Sigma_J)$ satisfies condition (I).

Let $\mathcal{I}$ be a proper ideal of $\mathcal{A} \rtimes_\rho \Lambda$. Put $J_{\mathcal{I}} := \mathcal{I} \cap \mathcal{A}$.

Lemma 4.6.

(i) *If $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I), then $J_{\mathcal{I}}$ is a proper ρ-invariant saturated ideal of $\mathcal{A}$. We then have $J_{\mathcal{I}_J} = J$.*

(ii) *If $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (II), then the C^*-symbolic crossed product $(\mathcal{A}/J_{\mathcal{I}}) \rtimes_{\rho^{J_{\mathcal{I}}}} \Lambda_{J_{\mathcal{I}}}$ is canonically isomorphic to the quotient algebra $\mathcal{A} \rtimes_\rho \Lambda/\mathcal{I}$.*

Proof. (i) By condition (I), $J_{\mathcal{I}}$ is a nozero ideal of $\mathcal{A}$, that is ρ-invariant. If $\rho_\alpha(x)$ belongs to $J_{\mathcal{I}}$ for all $\alpha \in \Sigma$, the identity $x = \sum_{\alpha\in\Sigma} S_\alpha \rho_\alpha(x) S_\alpha^*$ implies $x \in \mathcal{I}$, so that $J_{\mathcal{I}}$ is saturated. The equality $J_{\mathcal{I}_J} = J$ follows from Proposition 4.5.

(ii) Let $\pi_{\mathcal{I}} : \mathcal{A} \rtimes_\rho \Lambda \to \mathcal{A} \rtimes_\rho \Lambda/\mathcal{I}$ be the quotient map. Put $s_\alpha = \pi_{\mathcal{I}}(S_\alpha)$. Then $\alpha \in \Sigma_{J_{\mathcal{I}}}$ if and only if $s_\alpha \neq 0$. The following relations

$$\sum_{\beta\in\Sigma_{J_{\mathcal{I}}}} s_\beta s_\beta^* = 1, \qquad s_\alpha^* \pi_{\mathcal{I}}(x) s_\alpha = \pi_{\mathcal{I}}(\rho_\alpha(x)) \qquad \pi_{\mathcal{I}}(x) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi_{\mathcal{I}}(x)$$

for $x \in \mathcal{A}, \alpha \in \Sigma_{J_{\mathcal{I}}}$ hold. As $(\mathcal{A}/J_{\mathcal{I}}, \rho^{J_{\mathcal{I}}}, \Sigma_{J_{\mathcal{I}}})$ satisfies condition (I), the uniqueness of the C^*-symbolic crossed product $(\mathcal{A}/J_{\mathcal{I}}) \rtimes_{\rho^{J_{\mathcal{I}}}} \Lambda_{J_{\mathcal{I}}}$ yields a canonical isomorphism to the quotient algebra $\mathcal{A} \rtimes_\rho \Lambda/\mathcal{I}$. □

Let $\mathcal{I}_{J_{\mathcal{I}}}$ be the ideal of $\mathcal{A} \rtimes_\rho \Lambda$ generated by $J_{\mathcal{I}}$. Since $J_{\mathcal{I}} \subset \mathcal{I}$, the inclusion relation $\mathcal{I}_{J_{\mathcal{I}}} \subset \mathcal{I}$ is clear.

Lemma 4.7. *If $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (II), then there exists a canonical isomorphism from $(\mathcal{A}/J_{\mathcal{I}}) \rtimes_{\rho^{J_{\mathcal{I}}}} \Lambda_{J_{\mathcal{I}}}$ to the quotient algebra $\mathcal{A} \rtimes_\rho \Lambda/\mathcal{I}_{J_{\mathcal{I}}}$.*

Proof. Take an arbitrary element $x \in \mathcal{A}$. If $x \in J_{\mathcal{I}}$, then $x \in \mathcal{I}_{J_{\mathcal{I}}}$. Conversely $x \in \mathcal{I}_{J_{\mathcal{I}}}$ implies $x \in J_{\mathcal{I}}$ by Proposition 4.5. Hence $x \in J_{\mathcal{I}}$ if and only if $x \in \mathcal{I}_{J_{\mathcal{I}}}$. For $\alpha \in \Sigma$, we have $S_\alpha \in \mathcal{I}_{J_{\mathcal{I}}}$ if and only if $S_\alpha^* S_\alpha \in \mathcal{I}_{J_{\mathcal{I}}} \cap \mathcal{A}$. By Proposition 4.5, the latter condition is equivalent to the condition $\rho_\alpha(1) \in J_{\mathcal{I}}$. We know that $\alpha \notin \Sigma_{J_{\mathcal{I}}}$ if and only if $S_\alpha \in \mathcal{I}_{J_{\mathcal{I}}}$. By the uniqueness of the algebra $(\mathcal{A}/J_{\mathcal{I}}) \rtimes_{\rho^{J_{\mathcal{I}}}} \Lambda_{J_{\mathcal{I}}}$, it is canonically isomorphic to the quotient algebra $\mathcal{A} \rtimes_\rho \Lambda/\mathcal{I}_{J_{\mathcal{I}}}$. □

Proposition 4.8. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (II). For a proper ideal $\mathcal{I}$ of $\mathcal{A} \rtimes_\rho \Lambda$, let $\mathcal{I}_{J_{\mathcal{I}}}$ be the ideal of $\mathcal{A} \rtimes_\rho \Lambda$ generated by $J_{\mathcal{I}}$. Then we have $\mathcal{I}_{J_{\mathcal{I}}} = \mathcal{I}$.*

Proof. Since $\mathcal{I}_{J_{\mathcal{I}}} \subset \mathcal{I}$, there exists a quotient map $q_{\mathcal{I}} : \mathcal{A} \rtimes_\rho \Lambda/\mathcal{I}_{J_{\mathcal{I}}} \to \mathcal{A} \rtimes_\rho \Lambda/\mathcal{I}$. By Lemma 4.6, and Lemma 4.7, there exist canonical isomorphisms

$$\pi_1 : (\mathcal{A}/J_{\mathcal{I}}) \rtimes_{\rho^{J_{\mathcal{I}}}} \Lambda_{J_{\mathcal{I}}} \to \mathcal{A} \rtimes_\rho \Lambda/\mathcal{I}, \qquad \pi_2 : (\mathcal{A}/J_{\mathcal{I}}) \rtimes_{\rho^{J_{\mathcal{I}}}} \Lambda_{J_{\mathcal{I}}} \to \mathcal{A} \rtimes_\rho \Lambda/\mathcal{I}_{J_{\mathcal{I}}}.$$

Since $q_{\mathcal{I}} = \pi_1 \circ \pi_2^{-1}$, it is ismorphism so that we have $\mathcal{I}_{J_{\mathcal{I}}} = \mathcal{I}$. □

Therefore we have

Theorem 4.9. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (II). There exists an inclusion preserving bijective correspondence between ρ-invariant saturated ideals of $\mathcal{A}$ and ideals of $\mathcal{A} \rtimes_\rho \Lambda$, through the correspondences: $J \to \mathcal{I}_J$ and $J_{\mathcal{I}} \leftarrow \mathcal{I}$.*

5. Pure infiniteness

In this section we will show that the C^*-symbolic crossed product $\mathcal{A} \rtimes_\rho \Lambda$ is purely infinite if $(\mathcal{A}, \rho, \Sigma)$ satisfies some conditions.

Definition. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *central* if the projections $\{\rho_\mu(1) \mid \mu \in \Lambda^*\}$ contained in the center $Z_{\mathcal{A}}$ of $\mathcal{A}$. It is said to be *commutative* if $\mathcal{A}$ is commutative. Hence if $(\mathcal{A}, \rho, \Sigma)$ is central, the inequality $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ holds. Let $\mathcal{A}_\rho$ be the C^*-subalgebra of $\mathcal{A}$ generated by the projections $\rho_\mu(1), \mu \in \Lambda^*$.

Lemma 5.1. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is central. Then there exists a λ-graph system $\mathfrak{L}_\rho$ over Σ such that the presented subshift $\Lambda_{\mathfrak{L}_\rho}$ coincides with the subshift Λ presented by $(\mathcal{A}, \rho, \Sigma)$, and there exists a unital embedding of $\mathcal{O}_{\mathfrak{L}_\rho}$ into $\mathcal{A} \rtimes_\rho \Lambda$.*

Proof. Put $\mathcal{A}_{\rho,0} = \mathbb{C}$. For $l \in \mathbb{Z}_+$, we define the C^*-algebra $\mathcal{A}_{\rho,l+1}$ to be the C^*-subalgebra of $\mathcal{A}$ generated by the elements $\rho_\alpha(x)$ for $\alpha \in \Sigma, x \in \mathcal{A}_{\rho,l}$. Hence the C^*-algebra $\mathcal{A}_\rho$ is generated by $\cup_{l=0}^{\infty} \mathcal{A}_{\rho,l}$. Then $(\mathcal{A}_\rho, \rho, \Sigma)$ is a C^*-symbolic dynamical system such that $\mathcal{A}_\rho$ is commutative and AF, so that there exists a λ-graph system $\mathfrak{L}_\rho$ over Σ such that $\mathcal{A}_\rho = \mathcal{A}_{\mathfrak{L}_\rho}$. The presented subshift $\Lambda_{\mathfrak{L}_\rho}$ coincides with the subshift Λ. It is easy to see that there exists a unital embedding of $\mathcal{O}_{\mathfrak{L}_\rho}$ into $\mathcal{A} \rtimes_\rho \Lambda$ by their universalities. $\square$

In the rest of this section we assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I).

Definition. $(\mathcal{A}, \rho, \Sigma)$ is said to be *effective* if for $l \in \mathbb{Z}_+$ and a nonzero positive element $a \in \mathcal{A}_l$, there exist $K \in \mathbb{N}$ and a nonzero positive element $b \in \mathcal{A}_\rho$ such that

$$\sum_{\mu \in \Lambda^K} \rho_\mu(a) \geq b \tag{5.1}$$

where $\mathcal{A}_l$ is a C^*-subalgebra of $\mathcal{A}$ appearing in the definition of condition (I).

In what follows, we assume that $(\mathcal{A}, \rho, \Sigma)$ is effective, and central. Let $\mathfrak{L} = \mathfrak{L}_\rho$ be the λ-graph system associated to $(\mathcal{A}, \rho, \Sigma)$ as in Lemma 5.1. We further assume that the algebra $\mathcal{O}_{\mathfrak{L}}$ is simple, purely infinite. In [Ma2], [Ma3], conditions that the algebra $\mathcal{O}_{\mathfrak{L}}$ becomes simple, purely infinite is studied.

Lemma 5.2. *For $k \leq l \in \mathbb{Z}_+$ and a nonzero positive element $a \in \mathcal{F}_{\rho,l}^k$, there exists an element $V \in \mathcal{A} \rtimes_\rho \Lambda$ such that $VaV^* = 1$.*

Proof. An element $a \in \mathcal{F}_{\rho,l}^k$ is of the form $a = \sum_{\mu,\nu \in \Lambda^k} S_\mu a_{\mu,\nu} S_\nu^*$ for some $a_{\mu,\nu} \in \mathcal{A}_l$ such that $S_\mu^* a S_\nu = a_{\mu,\nu}$. Since a is a nonzero positive element, there exists $\xi \in \Lambda^k$ such that $S_\xi^* a S_\xi (= a_{\xi,\xi}) \neq 0$. As we are assuming that $(\mathcal{A}, \rho, \Sigma)$ is effective, there exists $K \in \mathbb{N}$ and a nonzero positive element $b \in \mathcal{A}_\rho$

$$\sum_{\mu \in \Lambda^K} \rho_\mu(S_\xi^* a S_\xi) \geq b.$$

Put $T = \sum_{\mu \in \Lambda^K} S_\mu \in \mathcal{A} \rtimes_\rho \Lambda$. One has $T^* S_\xi^* a S_\xi T \geq b$. Now $b \in \mathcal{A}_\rho \subset \mathcal{O}_{\mathfrak{L}}$ and $\mathcal{O}_{\mathfrak{L}}$ is simple, purely infinite. We may find $V_0 \in \mathcal{O}_{\mathfrak{L}}$ such that $V_0 b V_o^* = 1$ so that $V_0 T^* S_\xi^* a S_\xi T V_0^* \geq 1$. Hence there exists $V \in \mathcal{A} \rtimes_\rho \Lambda$ such that $VaV^* = 1$. $\square$

Lemma 5.3. *Keep the above situation. We may take $V \in \mathcal{A} \rtimes_\rho \Lambda$ in the preceding lemma such as $VaV^* = 1$ and $\|V\| < \|a\|^{-\frac{1}{2}} + \epsilon$ for a given $\epsilon > 0$.*

Proof. We may assume that $\|a\| = 1$ and there exists $p \in Sp(a)$ such that $0 < p < 1$. Take $0 < \epsilon < \frac{1}{2}$ such that $\epsilon < 1 - p$. Define a function $f \in C([0,1])$ by setting

$$f(t) = \begin{cases} 0 & (0 \le t \le 1-\epsilon) \\ 1 - \epsilon^{-1}(1-t) & 1-\epsilon < t \le 1 \end{cases}$$

Put $b = f(a)$, that is not invertible. By Lemma 5.2, there exists $V \in \mathcal{A} \rtimes_\rho \Lambda$ such that $VbV^* = 1$. We set $S = b^{\frac{1}{2}}V^*$ and $P = SS^*$. Then S is a proper isometry such that $P \le \|V\|b$. As $P \le E_a([1-\epsilon, 1])$, $E_a([1-\epsilon,1])$ is the spectral measure of a for the interval $[1-\epsilon, 1]$, one has $PaP \ge (1-\epsilon)P$. Put $D = S^*aS$ so that $D \ge S^*(1-\epsilon)PS = (1-\epsilon)1$. Hence D is invertible. Set $V_1 = D^{-\frac{1}{2}}S^*$. Then one sees that $V_1 a V_1^* = 1$ and $\|V_1\| < (1-\epsilon)^{-\frac{1}{2}} < 1 + \epsilon$. □

Let $\mathcal{E}_\rho : \mathcal{A} \rtimes_\rho \Lambda \to \mathcal{F}_\rho$ be the conditional expectation defined in Section 3.

Lemma 5.4. *For a nonzero $X \in \mathcal{A} \rtimes_\rho \Lambda$ and $\epsilon > 0$, there exists a projection $Q \in \mathcal{D}_\rho$ and a nonzero positive element $Z \in \mathcal{F}_{\rho,k}^l$ for some $k \le l$ such that*

$$\|QX^*XQ - Z\| < \epsilon, \qquad \|\mathcal{E}_\rho(X^*X)\| - \epsilon < \|Z\| < \|\mathcal{E}_\rho(X^*X)\| + \epsilon.$$

Proof. We may assume that $\|\mathcal{E}_\rho(X^*X)\| = 1$. Let $\mathcal{P}_\rho$ be the $*$-algebra generated algebraically by $S_\alpha, \alpha \in \Sigma$ and $x \in \mathcal{A}$. For any $0 < \epsilon < \frac{1}{4}$, find $0 \le Y \in \mathcal{P}_\rho$ such that $\|X^*X - Y\| < \frac{\epsilon}{2}$ so that $\|\mathcal{E}_\rho(Y)\| > 1 - \frac{\epsilon}{2}$. As in the discussion in [Ma3;Section 3], the element Y is expressed as

$$Y = \sum_{|\nu| \ge 1} Y_{-\nu} S_\nu^* + Y_0 + \sum_{|\mu| \ge 1} S_\mu Y_\mu \qquad \text{for some} \quad Y_{-\nu}, Y_0, Y_\mu \in \mathcal{F}_\rho \cap \mathcal{P}_\rho.$$

Take $k \le l$ large enough such that $Y_{-\nu}, Y_0, Y_\mu \in \mathcal{F}_{\rho,k}^l$ for all μ, ν in the above expression. Now $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I). Take a sequence $Q_k^l \in \mathcal{D}_\rho$ of projections as in Section 3. As $\mathcal{E}_\rho(Y) = Y_0$ and Q_k^l commutes with $\mathcal{F}_{\rho,k}^l$, it follows that by Lemma 3.5 (c), $Q_k^l Y Q_k^l = Q_k^l \mathcal{E}_\rho(Y) Q_k^l$. Since $Q_k^l \mathcal{E}_\rho(Y) Q_k^l \in \mathcal{F}_\rho$, there exists $0 \le Z \in \mathcal{F}_{\rho,k'}^{l'}$ for some $k' \le l'$ such that $\|Q_k^l \mathcal{E}_\rho(Y) Q_k^l - Z\| < \frac{\epsilon}{2}$. By Lemma 3.3, we note $\|Q_k^l \mathcal{E}_\rho(Y) Q_k^l\| = \|\mathcal{E}_\rho(Y)\|$ so that

$$\|Z\| \ge \|\mathcal{E}_\rho(Y)\| - \frac{\epsilon}{2} > 1 - \epsilon$$

and

$$\|Z\| < \|Q_k^l \mathcal{E}_\rho(Y) Q_k^l\| + \frac{\epsilon}{2} \le \|\mathcal{E}_\rho(X^*X)\| + \frac{\epsilon}{2} + \frac{\epsilon}{2} < 1 + \epsilon.$$

□

Therefore we have

Theorem 5.5. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is central, irreducible and satisfies condition (I). Let $\mathfrak{L}$ be the associated λ-graph system to $(\mathcal{A}, \rho, \Sigma)$. If $(\mathcal{A}, \rho, \Sigma)$ is effective and $\mathcal{O}_{\mathfrak{L}}$ is simple, purely infinite, then $\mathcal{A} \rtimes_\rho \Lambda$ is simple, purely infinite.*

Proof. It suffices to show that for any nonzero $X \in \mathcal{A} \rtimes_\rho \Lambda$, ther exist $A, B \in \mathcal{A} \rtimes_\rho \Lambda$ such that $AXB = 1$. By the previous lemma there exists a projection $Q \in \mathcal{D}_\rho$ and a nonzero positive element $Z \in \mathcal{F}_{\rho,k}^l$ for some $k \le l$ such that $\|QX^*XQ - Z\| < \epsilon$. We may assume that $\|\mathcal{E}_\rho(X^*X)\| = 1$ so that $1 - \epsilon < \|Z\| < 1 + \epsilon$. By Lemma 5.3, take an element $V \in \mathcal{A} \rtimes_\rho \Lambda$ such that

$$VZV^* = 1, \qquad \|V\| < \frac{1}{\sqrt{\|Z\|}} + \epsilon < \frac{1}{\sqrt{1-\epsilon}} + \epsilon.$$

It follows that

$$\|VQX^*XQV^* - 1\| < \|V\|^2 \|QX^*XQ - Z\| < (\frac{1}{\sqrt{1-\epsilon}} + \epsilon)^2 \cdot \epsilon.$$

We may take $\epsilon > 0$ small enough so that $\|VQX^*XQV^* - 1\| < 1$ and hence VQX^*XQV^* is invertible in $\mathcal{A} \rtimes_\rho \Lambda$. Thus we complete the proof. $\square$

6. Tensor products of C^*-symbolic dynamical systems

In this section, we will consider tensor products between C^*-symbolic dynamical systems and finite families of automorphisms of unital C^*-algebras. This construction yields interesting examples of C^*-symbolic dynamical systems beyond λ-graph systems, that will be studied in the following sections. Throughout this section, we fix a unital C^*-algebra $\mathcal{B}$ and a finite family of automorphisms $\alpha_i \in \mathrm{Aut}(\mathcal{B}), i = 1, \dots, N$ of $\mathcal{B}$. Tensor products $\otimes$ between C^*-algebras always mean the minimal C^*-tensor products $\otimes_{\min}$. We set $\Sigma = \{\alpha_1, \dots, \alpha_N\}$. Consider a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$.

Proposition 6.1. *For $\alpha_i \in \Sigma, i = 1, \dots, N$, define $\rho_{\alpha_i}^{\Sigma\otimes} \in \mathrm{End}(\mathcal{B} \otimes \mathcal{A})$ by setting*

$$\rho_{\alpha_i}^{\Sigma\otimes}(b \otimes a) = \alpha_i(b) \otimes \rho_{\alpha_i}(a) \qquad \text{for } b \in \mathcal{B}, a \in \mathcal{A}.$$

Then $(\mathcal{B} \otimes \mathcal{A}, \rho^{\Sigma\otimes}, \Sigma)$ becomes a C^*-symbolic dynamical system over Σ such that the presented subshift $\Lambda_{\rho^{\Sigma\otimes}}$ is the same as the subshift Λ_ρ presented by $(\mathcal{A}, \rho, \Sigma)$.

Proof. We will first prove that $(\mathcal{B} \otimes \mathcal{A}, \rho^{\Sigma\otimes}, \Sigma)$ is a C^*-symbolic dynamical system. Since $\{\rho_{\alpha_i}\}_{i=1}^N$ is essential, for $\epsilon > 0$, there exist $x_{i,j}, y_{i,j} \in \mathcal{A}$, $j = 1, \dots, n(i)$, $i = 1, \dots, N$ such that

$$\|\sum_{i=1}^N \sum_{j=1}^{n(i)} x_{i,j} \rho_{\alpha_i}(1) y_{i,j} - 1\| < \epsilon$$

so that we have

$$\|\sum_{i=1}^N \sum_{j=1}^{n(i)} (1 \otimes x_{i,j})(\rho_{\alpha_i}^{\Sigma\otimes}(1))(1 \otimes y_{i,j}) - 1\| < \epsilon.$$

Hence the closed ideal generated by $\{\rho_{\alpha_i}^{\Sigma\otimes}(1) : i = 1, \dots, N\}$ is all of $\mathcal{B} \otimes \mathcal{A}$, so that $\{\rho_{\alpha_i}^{\Sigma\otimes}\}_{i=1}^N$ is essential.

Since $\{\rho_{\alpha_i}\}_{i=1}^N$ is faithful on $\mathcal{A}$, the homomorphism $\xi_\rho : \mathcal{A} \longrightarrow \oplus_{i=1}^N \mathcal{A}_i$, where $\mathcal{A}_i = \mathcal{A}, i = 1, \dots, N$ defined by $\xi_\rho(a) = \oplus_{i=1}^N \rho_{\alpha_i}(a)$ is injective. Consider the homomorphisms:

$$\mathrm{id}_{\mathcal{B}} \otimes \xi_\rho : b \otimes a \in \mathcal{B} \otimes \mathcal{A} \to b \otimes \xi_\rho(a) \in \mathcal{B} \otimes \xi_\rho(\mathcal{A}),$$
$$\oplus_{i=1}^N (\alpha_i \otimes \mathrm{id}) : (b_i \otimes a_i)_{i=1}^N \in \oplus_{i=1}^N (\mathcal{B} \otimes \mathcal{A}_i) \to (\alpha_i(b_i) \otimes a_i)_{i=1}^N \in \oplus_{i=1}^N (\mathcal{B} \otimes \mathcal{A}_i).$$

Since $\mathcal{B} \otimes \xi_\rho(\mathcal{A})$ is a subalgebra of $\mathcal{B} \otimes (\oplus_{i=1}^N \mathcal{A}_i) = \oplus_{i=1}^N (\mathcal{B} \otimes \mathcal{A}_i)$ and both $\mathrm{id}_B \otimes \xi_\rho$ and $\oplus_{i=1}^N (\alpha_i \otimes \mathrm{id})$ are isomorphisms, the composition $\oplus_{i=1}^N (\alpha_i \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \xi_\rho)$ is isomorphic. Hence

$$\oplus_{i=1}^N \rho_{\alpha_i}^{\Sigma\otimes} = \oplus_{i=1}^N (\alpha_i \otimes \rho_{\alpha_i}) : \mathcal{B} \otimes \mathcal{A} \to \otimes_{i=1}^N (\mathcal{B} \otimes \mathcal{A}_i)$$

is injective. This implies that $\{\rho_{\alpha_i}^{\Sigma\otimes}\}_{i=1}^N$ is faithful.

By the equality

$$\rho_{\alpha_{i_n}}^{\Sigma\otimes} \circ \cdots \circ \rho_{\alpha_{i_1}}^{\Sigma\otimes}(1) = \rho_{\alpha_{i_n}} \circ \cdots \circ \rho_{\alpha_{i_1}}(1)$$

for $\alpha_{i_1}, \cdots, \alpha_{i_n} \in \Sigma$, the presented subshifts $\Lambda_{\rho^{\Sigma\otimes}}$ and Λ_ρ coincide. $\square$

We denote by Λ the presented subshift $\Lambda_\rho (= \Lambda_{\rho^{\Sigma\otimes}})$. Let S_{α_i} be the generating partial isometries of $\mathcal{A} \rtimes_\rho \Lambda$ satisfying $S_{\alpha_i}^* x S_{\alpha_i} = \rho_{\alpha_i}(x)$ for $x \in \mathcal{A}, i = 1, \dots, N$, and $\widetilde{S}_{\alpha_i}$ those of $(\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda$ satisfying $\widetilde{S}_{\alpha_i}^* y \widetilde{S}_{\alpha_i} = \rho_{\alpha_i}^{\Sigma\otimes}(y)$ for $y \in \mathcal{B} \otimes \mathcal{A}, i = 1, \dots, N$.

Proposition 6.2. *There exists a unital embedding $\tilde{\iota}$ of $\mathcal{A} \rtimes_\rho \Lambda$ into $(\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda$ in a canonical way.*

Proof. Define the injective homomorphism $\iota : \mathcal{A} \to \mathcal{B} \otimes \mathcal{A}$ by setting $\iota(a) = 1 \otimes a$ for $a \in A$. Since the equality $\widetilde{S}_{\alpha_i}^* \iota(a) \widetilde{S}_{\alpha_i} = \iota(\rho_{\alpha_i}(a))$ for $a \in \mathcal{A}, i = 1, \dots, N$ holds, there exists a homomorphism $\tilde{\iota}$ from $\mathcal{A} \rtimes_\rho \Lambda$ to $(\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda$ satisfying $\tilde{\iota}(a) = 1 \otimes a, \tilde{\iota}(S_{\alpha_i}) = \widetilde{S}_{\alpha_i}$ for $a \in \mathcal{A}, i = 1, \dots, N$ by the universality of $\mathcal{A} \rtimes_\rho \Lambda$. Let $\mathcal{E}_\rho : \mathcal{A} \rtimes_\rho \Lambda \to \mathcal{F}_\rho$ and $\mathcal{E}_{\rho^{\Sigma\otimes}} : (\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda \to \mathcal{F}_{\rho^{\Sigma\otimes}}$ be the canonical conditional expectations respectively. Define the C^*-subalgebras $\mathcal{F}_{(\mathbb{C}\otimes\mathcal{A}, \rho^{\Sigma\otimes})} \subset (\mathbb{C} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda$ of $(\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda$ by setting

$$(\mathbb{C} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda = C^*(1 \otimes a, \widetilde{S}_{\alpha_i} : a \in \mathcal{A}, i = 1, \dots, N),$$
$$\mathcal{F}_{(\mathbb{C}\otimes\mathcal{A}, \rho^{\Sigma\otimes})} = C^*(\widetilde{S}_\mu (1 \otimes a) \widetilde{S}_\nu^* : a \in \mathcal{A},\ \mu, \nu \in \Lambda^*, |\mu| = |\nu|).$$

The diagrams

$$\begin{array}{ccccc} \mathcal{A} \rtimes_\rho \Lambda & \xrightarrow{\tilde{\iota}} & (\mathbb{C} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda & \hookrightarrow & (\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda \\ \mathcal{E}_\rho \Big\downarrow & & \Big\downarrow \mathcal{E}_{\rho^{\Sigma\otimes}}|_{\mathbb{C}\otimes\mathcal{A}} & & \Big\downarrow \mathcal{E}_{\rho^{\Sigma\otimes}} \\ \mathcal{F}_\rho & \xrightarrow{\tilde{\iota}|_{\mathcal{F}_\rho}} & \mathcal{F}_{(\mathbb{C}\otimes\mathcal{A}, \rho^{\Sigma\otimes})} & \hookrightarrow & \mathcal{F}_{\rho^{\Sigma\otimes}} \end{array}$$

are commutative. Since $\iota : \mathcal{A} \to \mathbb{C} \otimes \mathcal{A}$ is isomorphic, so is the restriction $\tilde{\iota}|_{\mathcal{F}_\rho} : \mathcal{F}_\rho \to \mathcal{F}_{(\mathbb{C}\otimes\mathcal{A}, \rho^{\Sigma\otimes})}$ of $\mathcal{F}_\rho$. One indeed sees that the condition $S_\mu a S_\nu^* \neq 0$ for some $a \in \mathcal{A}, |\mu| = |\nu|$ implies $\widetilde{S}_\mu (1 \otimes a) \widetilde{S}_\nu^* \neq 0$ because of the equality $\iota(\rho_\mu(1) a \rho_\nu(1)) =$

$\widetilde{S}_\mu^*\widetilde{S}_\mu(1\otimes a)\widetilde{S}_\nu^*\widetilde{S}_\nu$. For $\sum_{\mu,\nu\in\Lambda^k} S_\mu a_{\mu,\nu} S_\nu^* \in \mathcal{F}_\rho$, suppose that $\tilde{\imath}(\sum_{\mu,\nu\in\Lambda^k} S_\mu a_{\mu,\nu} S_\nu^*) = 0$. It follows that for any $\xi,\eta\in\Lambda^k$,

$$0 = \widetilde{S}_\xi^*(\sum_{\mu,\nu\in\Lambda^k} \widetilde{S}_\mu(1\otimes a_{\mu,\nu})\widetilde{S}_\nu^*)\widetilde{S}_\eta = \widetilde{S}_\xi^*\widetilde{S}_\xi(1\otimes a_{\xi,\eta})\widetilde{S}_\eta^*\widetilde{S}_\eta$$

so that $0 = \widetilde{S}_\xi(1\otimes a_{\xi,\eta})\widetilde{S}_\eta^*$, and hence $S_\xi a_{\xi,\eta} S_\nu^* = 0$. This implies that $\tilde{\imath}|_{\mathcal{F}_{\rho^k}} : \mathcal{F}_\rho^k \to \mathcal{F}^k_{(\mathbb{C}\otimes\mathcal{A},\rho^{\Sigma\otimes})}$ is injective and so is $\tilde{\imath}|_{\mathcal{F}_\rho} : \mathcal{F}_\rho \to \mathcal{F}_{(\mathbb{C}\otimes\mathcal{A},\rho^{\Sigma\otimes})}$. Therefore by using a routine argument, one concludes that $\tilde{\imath} : \mathcal{A}\rtimes_\rho\Lambda \to (\mathbb{C}\otimes\mathcal{A})\rtimes_{\rho^{\Sigma\otimes}}\Lambda$ is injective and hence isomorphic. $\square$

Let us prove that $(\mathcal{B}\otimes\mathcal{A},\rho^{\Sigma\otimes},\Sigma)$ satisfies condition (I) if $(\mathcal{A},\rho,\Sigma)$ satisfies condition (I). The result will be used in the following sections. We set the C^*-subalgebras $\mathcal{D}_{(\mathbb{C}\otimes\mathcal{A},\rho^{\Sigma\otimes})} \subset \mathcal{D}_{\rho^{\Sigma\otimes}}$ of $\mathcal{F}_{\rho^{\Sigma\otimes}}$ by setting

$$\begin{aligned}\mathcal{D}_{\rho^{\Sigma\otimes}} =& C^*(\widetilde{S}_\mu x\widetilde{S}_\mu^* : \mu\in\Lambda^*, x\in\mathcal{B}\otimes\mathcal{A}),\\ \mathcal{D}_{(\mathbb{C}\otimes\mathcal{A},\rho^{\Sigma\otimes})} =& C^*(\widetilde{S}_\mu(1\otimes a)\widetilde{S}_\mu^* : \mu\in\Lambda^*, a\in\mathcal{A}).\end{aligned}$$

We may identify the subalgebra $\mathcal{D}_\rho$ of $\mathcal{F}_\rho$ with the subalgebra $\mathcal{D}_{(\mathbb{C}\otimes\mathcal{A},\rho^{\Sigma\otimes})}$ of $\mathcal{F}_{(\mathbb{C}\otimes\mathcal{A},\rho^{\Sigma\otimes})}$ through the map $\tilde{\imath}$ as in the preceding proposition.

Let $\varphi\in\mathcal{B}^*$ be a faithful state on $\mathcal{B}$. It is well-known that there exists a faithful projection $\Theta_\varphi : \mathcal{B}\otimes\mathcal{A}\to\mathcal{A}$ of norm one satisfying $\Theta_\varphi(b\otimes a) = \varphi(b)a$ for $b\otimes a\in\mathcal{B}\otimes\mathcal{A}$.

Lemma 6.3. *Let $\varphi\in\mathcal{B}^*$ be a faithful state on $\mathcal{B}$ satisfying $\varphi\circ\alpha_i = \varphi, i = 1,\dots,N$. The projection $\Theta_\varphi : \mathcal{B}\otimes\mathcal{A}\to\mathcal{A}$ of norm one can be extended to a projection of norm one $\Theta_\mathcal{D} : \mathcal{D}_{\rho^{\Sigma\otimes}} \to \mathcal{D}_\rho$ such that $\Theta_\mathcal{D}(x) = x$ for $x\in\mathcal{D}_\rho$.*

Proof. For $k\in\mathbb{N}$, define the C^*-subalgebras $\mathcal{D}_\rho^k$ of $\mathcal{D}_\rho$ and $\mathcal{D}^k_{\rho^{\Sigma\otimes}}$ of $\mathcal{D}_{\rho^{\Sigma\otimes}}$ by setting

$$\begin{aligned}\mathcal{D}_\rho^k =& C^*(S_\mu a S_\mu^* : \mu\in\Lambda^k, a\in\mathcal{A}),\\ \mathcal{D}^k_{\rho^{\Sigma\otimes}} =& C^*(\widetilde{S}_\mu x\widetilde{S}_\mu^* : \mu\in\Lambda^k, x\in\mathcal{B}\otimes\mathcal{A}).\end{aligned}$$

For $x_\mu\in\mathcal{B}\otimes\mathcal{A}, \xi\in\Lambda^k$, the identities

$$\begin{aligned}\Theta_\varphi(\widetilde{S}_\xi^*(\sum_{\mu\in\Lambda^k}\widetilde{S}_\mu x_\mu\widetilde{S}_\mu^*)\widetilde{S}_\xi) &= \Theta_\varphi((1\otimes\rho_\xi(1))x_\xi(1\otimes\rho_\xi(1))\\ &= \rho_\xi(1)\Theta_\varphi(x_\xi)\rho_\xi(1) = S_\xi^*S_\xi\Theta_\varphi(x_\xi)S_\xi^*S_\xi\end{aligned}$$

hold, so that the map defined by $\Theta_\mathcal{D}^k : \mathcal{D}^k_{\rho^{\Sigma\otimes}} \to \mathcal{D}_\rho^k$

$$\Theta_\mathcal{D}^k(\sum_{\mu\in\Lambda^k}\widetilde{S}_\mu x_\mu\widetilde{S}_\mu^*) = \sum_{\mu\in\Lambda^k} S_\mu\Theta_\varphi(x_\mu)S_\mu^*.$$

is well-defined for each $k\in\mathbb{Z}_+$. We will next see the restriction of $\Theta_\mathcal{D}^{k+1}$ to $\mathcal{D}^k_{\rho^{\Sigma\otimes}}$ coincides with $\Theta_\mathcal{D}^k$. Since $\sum_{\mu\in\Lambda^k}\widetilde{S}_\mu x_\mu\widetilde{S}_\mu^*\in\mathcal{D}^k_{\rho^{\Sigma\otimes}}$ is written as $\sum_{\mu\in\Lambda^k}\sum_{i=1}^N\widetilde{S}_\mu\widetilde{S}_{\alpha_i}\widetilde{S}_{\alpha_i}^* x_\mu\widetilde{S}_{\alpha_i}\widetilde{S}_{\alpha_i}^*\widetilde{S}_\mu^*\in\mathcal{D}^{k+1}_{\rho^{\Sigma\otimes}}$, it follows that

$$\Theta_\mathcal{D}^{k+1}(\widetilde{S}_\mu x_\mu\widetilde{S}_\mu^*) = \sum_{i=1}^N\Theta_\mathcal{D}^{k+1}(\widetilde{S}_{\mu\alpha_i}\rho_{\alpha_i}^{\Sigma\otimes}(x_\mu)\widetilde{S}_{\mu\alpha_i}^* = \sum_{i=1}^N S_{\mu\alpha_i}\Theta_\varphi(\rho_{\alpha_i}^{\Sigma\otimes}(x_\mu))S_{\mu\alpha_i}^*.$$

As the state φ is α_i-invariant for $i = 1, \dots, N$, one has for $\sum_j b_j \otimes a_j \in \mathcal{B} \otimes \mathcal{A}$,

$$\Theta_\varphi(\rho^{\Sigma\otimes}_{\alpha_i}(\sum_j b_j \otimes a_j)) = \sum_j \varphi(\alpha_i(b_j))\rho_{\alpha_i}(a_j) = \sum_j \varphi(b_j)\rho_{\alpha_i}(a_j)$$
$$= \rho_{\alpha_i}(\sum_j \varphi(b_j)a_j) = S^*_{\alpha_i}\Theta_\varphi(\sum_j b_j \otimes a_j)S_{\alpha_i}$$

so that $\Theta_\varphi(\rho^{\Sigma\otimes}_{\alpha_i}(x_\mu)) = S^*_{\alpha_i}\Theta_\varphi(x_\mu)S_{\alpha_i}$ for $x_\mu \in \mathcal{B} \otimes \mathcal{A}$. It then follows that

$$\Theta^{k+1}_{\mathcal{D}}(\widetilde{S}_\mu x_\mu \widetilde{S}^*_\mu) = \sum_{i=1}^N S_{\mu\alpha_i}S^*_{\alpha_i}\Theta_\varphi(x_\mu)S_{\alpha_i}S^*_{\mu\alpha_i} = S_\mu\Theta_\varphi(x_\mu)S^*_\mu = \Theta^k_{\mathcal{D}}(\widetilde{S}_\mu x_\mu \widetilde{S}^*_\mu).$$

Therefore the sequence $\{\Theta^k_{\mathcal{D}}\}_{k=1}^\infty$ defines a projection from $\mathcal{D}_{\rho^{\Sigma\otimes}}$ onto $\mathcal{D}_\rho$, which we denote by $\Theta_{\mathcal{D}}$. ☐

Lemma 6.4. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is central. Then $\widetilde{S}_\mu(1 \otimes a)\widetilde{S}^*_\mu$ commutes with $b \otimes 1$ for $a \in \mathcal{A}, \mu \in \Lambda^*$ and $b \in \mathcal{B}$.*

Proof. Since $(1 \otimes a)\rho^{\Sigma\otimes}_\mu(b \otimes 1) = \rho^{\Sigma\otimes}_\mu(b \otimes 1)(1 \otimes a)$, it follows that

$$\begin{aligned}&\widetilde{S}_\mu(1 \otimes a)\widetilde{S}^*_\mu(b \otimes 1)\\ =&\widetilde{S}_\mu(1 \otimes a)\rho^{\Sigma\otimes}_\mu(b \otimes 1)\widetilde{S}^*_\mu = \widetilde{S}_\mu\widetilde{S}^*_\mu(b \otimes 1)\widetilde{S}_\mu(1 \otimes a)\widetilde{S}^*_\mu = (b \otimes 1)\widetilde{S}_\mu(1 \otimes a)\widetilde{S}^*_\mu.\end{aligned}$$

☐

Theorem 6.5. *Assume that there exists a faithful state φ on $\mathcal{B}$ invariant under $\alpha_i \in \mathrm{Aut}(\mathcal{B}), i = 1, \dots, N$. Suppose that $(\mathcal{A}, \rho, \Sigma)$ is central. If $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I), then $(\mathcal{B} \otimes \mathcal{A}, \rho^{\Sigma\otimes}, \Sigma)$ satisfies condition (I) and is central.*

Proof. Since $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I), there exists a increasing sequence $\mathcal{A}_l, l \in \mathbb{Z}_+$ of C^*-subalgebras of $\mathcal{A}$ and a projection $q^l_k \in \mathcal{D}_\rho \cap \mathcal{A}_l{}'$ with $l \geq k$ satisfying the conditions of condition (I). We set $(\mathcal{B} \otimes \mathcal{A})_l = \mathcal{B} \otimes \mathcal{A}_l, l \in \mathbb{Z}_+$. Then the conditions $\overline{\cup_{l\in\mathbb{N}}(\mathcal{B} \otimes \mathcal{A})_l} = \mathcal{B} \otimes \mathcal{A}$ and $\rho^{\Sigma\otimes}_{\alpha_i}((\mathcal{B} \otimes \mathcal{A})_l) \subset (\mathcal{B} \otimes \mathcal{A})_{l+1}$ are easy to veryfy. Let $\tilde{\iota} : \mathcal{A} \rtimes_\rho \Lambda \hookrightarrow (\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda$ be the embedding in Proposition 6.2. Put $\tilde{q}^l_k = \tilde{\iota}(q^l_k) \in \mathcal{D}_{\rho^{\Sigma\otimes}}$ for $l \geq k$. By the preceding lemma, one sees that $\tilde{q}^l_k \in \mathcal{D}_{\rho^{\Sigma\otimes}} \cap ((\mathcal{B} \otimes \mathcal{A})_l)'$. We will show that $\tilde{q}^l_k x \neq 0$ for $0 \neq x \in (\mathcal{B} \otimes \mathcal{A})_l$. As $xx^* \in \mathcal{B} \otimes \mathcal{A}_l$, one has $\Theta_{\mathcal{D}}(xx^*) = \Theta_\varphi(xx^*) \in \mathcal{A}_l$. Hence $q^l_k\Theta_\varphi(xx^*) \neq 0$. By the equality $\Theta_{\mathcal{D}}(\tilde{q}^l_k xx^* \tilde{q}^l_k) = q^l_k\Theta_\varphi(xx^*)q^l_k$, one obtains $\tilde{q}^l_k x \neq 0$. Let $\tilde{\phi}_{\rho^{\Sigma\otimes}}(X) = \sum_{i=1}^N \widetilde{S}_{\alpha_i} X \widetilde{S}^*_{\alpha_i}$ for $X \in \mathcal{D}_{\rho^{\Sigma\otimes}}$. One has

$$\tilde{q}^l_k \tilde{\phi}^m_{\rho^{\Sigma\otimes}}(\tilde{q}^l_k) = \tilde{\iota}(q^l_k \phi^m_\rho(q^l_k)) = 0 \qquad \text{for all } m = 1, 2, \dots, k.$$

Thus $(\mathcal{B} \otimes \mathcal{A}, \rho^{\Sigma\otimes}, \Sigma)$ satisfies condition (I). If $(\mathcal{A}, \rho, \Sigma)$ is central, the projections $1 \otimes \rho_\mu(1)$ for $\mu \in \Lambda^*$ commute with $\mathcal{B} \otimes \mathcal{A}$, so that $(\mathcal{B} \otimes \mathcal{A}, \rho^{\Sigma\otimes}, \Sigma)$ is central. ☐

We will study structure of the fixed point algebra $\mathcal{F}_{\rho^{\Sigma\otimes}}$ of $(\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma\otimes}} \Lambda$ under the gauge action $\widehat{\rho^{\Sigma\otimes}}$. Recall that $\mathcal{F}_\rho$ denote the fixed point algebra of $\mathcal{A} \rtimes_\rho \Lambda$ under the gauge action $\hat{\rho}$. Recall that for $k \in \mathbb{Z}_+$ the C^*-subalgebras $\mathcal{F}^k_\rho$ of $\mathcal{F}_\rho$ and $\mathcal{F}^k_{\rho^{\Sigma\otimes}}$ of $\mathcal{F}_{\rho^{\Sigma\otimes}}$ are generated by $S_\mu a S^*_\nu$ for $\mu, \nu \in \Lambda^k, a \in \mathcal{A}$ and $\widetilde{S}_\mu x \widetilde{S}^*_\nu$ for $\mu, \nu \in \Lambda^k, x \in \mathcal{B} \otimes \mathcal{A}$ respectively. Then we have

Lemma 6.6. *The map $\Phi^k : \widetilde{S}_\mu(b\otimes a)\widetilde{S}_\nu^* \to b\otimes S_\mu a S_\nu^*$ for $b\otimes a \in \mathcal{B}\otimes\mathcal{A},\ \mu,\nu\in\Lambda^k$ extends to an isomorphism from $\mathcal{F}^k_{\rho^{\Sigma\otimes}}$ to $\mathcal{B}\otimes\mathcal{F}^k_\rho$.*

Proof. For $Y = \sum_{\mu,\nu\in\Lambda^k}\widetilde{S}_\mu(\sum_{j=1}^n b_j\otimes a_j)\widetilde{S}_\nu^* \in \mathcal{F}^k_{\rho^{\Sigma\otimes}}$, put

$$\Phi^k(Y) = \sum_{j=1}^n (b_j \otimes \sum_{\mu,\nu\in\Lambda^k} S_\mu a_j S_\nu^*) \in \mathcal{B}\otimes\mathcal{F}^k_\rho.$$

It follows that for $\xi,\eta\in\Lambda^k$

$$\widetilde{S}_\xi^* Y \widetilde{S}_\eta = \widetilde{S}_\xi^*\widetilde{S}_\xi(\sum_{j=1}^n b_j\otimes a_j)\widetilde{S}_\eta^*\widetilde{S}_\eta = \sum_{j=1}^n b_j \otimes S_\xi^* S_\xi a_j S_\eta^* S_\eta = (1\otimes S_\xi^*)\Phi^k(Y)(1\otimes S_\eta)$$

Hence $Y=0$ if and only if $\Phi^k(Y)=0$. As Φ^k is a homomorphism from $\mathcal{F}^k_{\rho^{\Sigma\otimes}}$ to $\mathcal{B}\otimes\mathcal{F}^k_\rho$, it yields an isomorphism. □

The following lemma is straightforward.

Lemma 6.7. *Let $\alpha\otimes\iota^k_\rho : \mathcal{B}\otimes\mathcal{F}^k_\rho \to \mathcal{B}\otimes\mathcal{F}^{k+1}_\rho$ be the homomorphism defined by*

$$(\alpha\otimes\iota^k_\rho)(b\otimes S_\mu a S_\nu^*) = \sum_{i=1}^n \alpha_i(b)\otimes S_{\mu\alpha_i}\rho_{\alpha_i}(a)S^*_{\nu\alpha_i} \qquad \text{for } b\otimes a\in\mathcal{B}\otimes\mathcal{A},\ \mu,\nu\in\Lambda^k.$$

Then the diagram

$$\begin{array}{ccc}
\mathcal{F}^k_{\rho^{\Sigma\otimes}} & \xrightarrow{\iota^k_{\rho^{\Sigma\otimes}}} & \mathcal{F}^{k+1}_{\rho^{\Sigma\otimes}} \\
{\scriptstyle\Phi^k}\downarrow & & \downarrow{\scriptstyle\Phi^{k+1}} \\
\mathcal{B}\otimes\mathcal{F}^k_\rho & \xrightarrow[\alpha\otimes\iota^k_\rho]{} & \mathcal{B}\otimes\mathcal{F}^{k+1}_\rho
\end{array}$$

is commutative, where $\iota^k_{\rho^{\Sigma\otimes}} : \mathcal{F}^k_{\rho^{\Sigma\otimes}} \to \mathcal{F}^{k+1}_{\rho^{\Sigma\otimes}}$ denotes the natural inclusion.

Hence we have

Proposition 6.8. *The C^*-algebra $\mathcal{F}_{\rho^{\Sigma\otimes}}$ is the inductive limit*

$$\mathcal{B}\otimes\mathcal{F}^1_\rho \xrightarrow{\alpha\otimes\iota^1_\rho} \mathcal{B}\otimes\mathcal{F}^2_\rho \xrightarrow{\alpha\otimes\iota^2_\rho} \mathcal{B}\otimes\mathcal{F}^3_\rho \xrightarrow{\alpha\otimes\iota^3_\rho} \cdots.$$

Let $\mathcal{B} = C(X)$ be the commutative C^*-algebra of all continuous functins on a compact Hausdorff space X with a finite family $h_1,\dots,h_N$ of homeomorphisms on X. Define $\alpha_i \in \mathrm{Aut}(C(X)), i=1,\dots,N$ by $\alpha_i(f)(t) = f(h_i(t))$ for $f\in C(X), t\in X$. Put $\Sigma = \{\alpha_1,\dots,\alpha_N\}$ Take $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ for a λ-graph system $\mathfrak{L}$ over Σ as $(\mathcal{A},\rho,\Sigma)$. Then the above C^*-algebra $\mathcal{F}_{\rho^{\Sigma\otimes}}$ is an AH-algebra. If in particular $X=\mathbb{T}$, the algbera is an A$\mathbb{T}$-algebra. We will study these examples in the following sections.

7. C^*-SYMBOLIC DYNAMICAL SYSTEMS FROM HOMEOMORPHISMS AND GRAPHS

Let $h_1, \dots, h_N$ be a finite family of homemorphisms on a compact Hausdorff space X. Put $\Sigma = \{h_1, \dots, h_N\}$. Let $\mathcal{G}$ be a left-resolving finite labeled graph (G, λ) over Σ with underlying finite directed graph G and labeling map $\lambda : E \to \Sigma$. We denote by $G = (V, E)$, where $V = \{v_1, \dots, v_{N_0}\}$ is the finite set of its vertices and $E = \{e_1, \dots, e_{N_1}\}$ is the finite set of its directed edges. As in the begining of Section 2, we have a C^*-symbolic dynamical system $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$. Identify the homeomorphisms h_i with the induced automorphisms α_i on $C(X)$. By Proposition 6.1, the tensor product $(C(X) \otimes \mathcal{A}_{\mathcal{G}}, (\rho^{\mathcal{G}})^{\Sigma\otimes}, \Sigma)$ of C^*-symbolic dynamical system is defined. Put $X_i = X, i = 1, \dots, N_0$ and

$$\mathcal{A}_{\mathcal{G},X} = C(X) \otimes \mathcal{A}_{\mathcal{G}} = C(\sqcup_{i=1}^{N_0} X_i), \qquad \rho^{\mathcal{G},X} = (\rho^{\mathcal{G}})^{\Sigma\otimes}.$$

We will study the C^*-symbolic dynamical system $(\mathcal{A}_{\mathcal{G},X}, \rho^{\mathcal{G},X}, \Sigma)$. Note that the presented subshift $\Lambda_{\rho^{\mathcal{G},X}}$ is the sofic shift $\Lambda_{\mathcal{G}}$ presented by the labeled graph $\mathcal{G}$.

For $u, v \in V$, let $H_n(u,v)$ be the set $(f_1, \dots, f_n)$ of n-edges of the graph $\mathcal{G}$ satisfying $s(f_1) = u, t(f_i) = s(f_{i+1}), i = 1, \dots, n-1$, and $t(f_n) = v$. We set

$$H_n(u) = \cup_{v\in V} H_n(u,v), \qquad H_{\mathcal{G}}^n = \cup_{u \in V} H_n(u), \qquad H_{\mathcal{G}} = \cup_{n=1}^{\infty} H_{\mathcal{G}}^n.$$

Then $\gamma = (f_1, \dots, f_n) \in H_n(v_i, v_j)$ yields a homeomorphism $\lambda(\gamma)$ from X_i to X_j by setting

$$\lambda(\gamma)(x) = \lambda(f_n) \circ \cdots \circ \lambda(f_1)(x) \qquad \text{for } x \in X_i.$$

For $x \in X_k$ with $k \ne i$, $\lambda(\gamma)(x)$ is not defined. We set for $x \in X_i$

$$orb_n(x) = \cup\{\lambda(\gamma)(x) \mid \gamma \in H_n(v_i)\} \subset \sqcup_{j=1}^{N_0} X_j, \qquad orb(x) = \cup_{n=0}^{\infty} orb_n(x),$$

where $orb_0(x) = \{x\}$.

Definition. A family $(h_1, \dots, h_N)$ of homeomorphisms on X is called $\mathcal{G}$-*minimal* if for any $x \in \sqcup_{j=1}^{N_0} X_j$, the orbit $orb(x)$ is dense in $\sqcup_{j=1}^{N_0} X_j$.

Lemma 7.1. *The following conditions are equivalent:*

(i) $(h_1, \dots, h_N)$ *is* $\mathcal{G}$*-minimal;*

(ii) *There exists no proper closed subset* $F \subset \sqcup_{j=1}^{N_0} X_j$ *such that* $\lambda(e_i)(F) \subset F$ *for all* $i = 1, \dots, N_1$*;*

(iii) *There exists no proper closed subset* $F \subset \sqcup_{j=1}^{N_0} X_j$ *such that* $\cup_{i=1}^{N_1} \lambda(e_i)(F) = F$.

Proof. (i)⇒(ii) If there exists a closed subset $F \subset \sqcup_{j=1}^{N_0} X_j$ such that $\lambda(e_i)(F) \subset F$ for all $i = 1, \dots, N_1$, take $x \in F \cap X_j$ for some j. Then $orb(x)$ is not dense in $\sqcup_{j=1}^{N_0} X_j$.

(ii)⇒(i) For $x \in \sqcup_{j=1}^{N_0} X_j$, let F be the closure of $orb(x)$. Then we have $\lambda(e_i)(F) \subset F$ for all $i = 1, \dots, N_1$, and hence $F = \sqcup_{j=1}^{N_0} X_j$.

(ii)⇒(iii) This implication is trivial.

(iii)⇒(ii) Suppose that there exists a closed subset $F \subset \sqcup_{j=1}^{N_0} X_j$ such that $\lambda(e_i)(F) \subset F$ for all $i = 1, \dots, N_1$. Put $\widetilde{F}_n = \cup_{\lambda(\gamma) \in H_{\mathcal{G}}^n} \lambda(\gamma)(F)$ a closed subset of F. Since $\widetilde{F}_{n+1} \subset \widetilde{F}_n$ and $\sqcup_{j=1}^{N_0} X_j$ is compact, the set $E := \cap_{n=1}^{\infty} \widetilde{F}_n$ is a nonempty

closed subset of $\sqcup_{j=1}^{N_0} X_j$. Since $\cup_{i=1}^{N_1}\lambda(e_i)(\widetilde{F}_n) = \widetilde{F}_{n+1}$, one has $\cup_{i=1}^{N}\lambda(e_i)(E) \subset E$. On the other hand, take $s(i) = 1,\dots,N_0$ such that $v_{s(i)} = s(e_i)$. Then we have

$$\begin{aligned}&\cap_{n=1}^{\infty} \lambda(e_i)(\widetilde{F}_n) = \cap_{n=1}^{\infty} \sqcup_{j=1}^{N_0} \lambda(e_i)(\widetilde{F}_n \cap X_j) = \cap_{n=1}^{\infty}\lambda(e_i)(\widetilde{F}_n \cap X_{s(i)})\\ \subset &\sqcup_{j=1}^{N_0} \cap_{n=1}^{\infty}\lambda(e_i)(\widetilde{F}_n \cap X_j) = \lambda(e_i)(E).\end{aligned}$$

For $x \in \cap_{n=1}^{\infty} \cup_{i=1}^{N_1} \lambda(e_i)(\widetilde{F}_n)$ and $n \in \mathbb{N}$, there exits $i_n = 1,\dots,N_1$ such that $x \in \lambda(e_{i_n})(\widetilde{F}_n)$. Find $i(x) = 1,\dots,N_1$ such that $i(x)$ appears in $\{i_n \mid n \in \mathbb{N}\}$ infinitely many times. Since $\widetilde{F}_n, n \in \mathbb{N}$ are decreasing subsets, one has $x \in \lambda(e_{i(x)})(\widetilde{F}_n)$ for all $n \in \mathbb{N}$. Hence $x \in \cup_{i=1}^{N_1} \cap_{n=1}^{\infty} \lambda(e_i)(\widetilde{F}_n)$ so that we have $\cup_{i=1}^{N_1} \cap_{n=1}^{\infty} \lambda(e_i)(\widetilde{F}_n) \supset \cap_{n=1}^{\infty} \cup_{i=1}^{N_1} \lambda(e_i)(\widetilde{F}_n)$. Thus we have

$$\cup_{i=1}^{N_1}\lambda(e_i)(E) \supset \cup_{i=1}^{N_1} \cap_{n=1}^{\infty} \lambda(e_i)(\widetilde{F}_n) \supset \cap_{n=1}^{\infty} \cup_{i=1}^{N_1} \lambda(e_i)(\widetilde{F}_n) = \cap_{n=1}^{\infty}\widetilde{F}_{n+1} = E.$$

□

The following lemma is direct.

Lemma 7.2. *Let J be an ideal of $\mathcal{A}_{\mathcal{G},X}$. Denote by $F \subset \sqcup_{j=1}^{N_0} X_j$ the closed subset such that $J = \{f \in C(\sqcup_{j=1}^{N_0} X_j) \mid f(x) = 0$ for $x \in F\}$. Then we have*

(i) *J is a $\rho^{\mathcal{G},X}$-invariant ideal of $\mathcal{A}_{\mathcal{G},X}$ if and only if $\lambda(e_i)(F) \subset F$ for all $i = 1,\dots,N_1$.*
(ii) *J is a saturated ideal of $\mathcal{A}_{\mathcal{G},X}$ if and only if $\cup_{i=1}^{N}\lambda(e_i)(F) \supset F$.*
(iii) *J is a $\rho^{\mathcal{G},X}$-invariant saturated ideal of $\mathcal{A}_{\mathcal{G},X}$ if and only if $\cup_{i=1}^{N}\lambda(e_i)(F) = F$.*

Hence we have

Lemma 7.3. *The following conditions are equivalent:*

(i) *$(h_1,\dots,h_N)$ is $\mathcal{G}$-minimal;*
(ii) *There exists no proper $\rho^{\mathcal{G},X}$-invariant ideal of $\mathcal{A}_{\mathcal{G},X}$;*
(iii) *There exists no proper $\rho^{\mathcal{G},X}$-invariant saturated ideal of $\mathcal{A}_{\mathcal{G},X}$.*

A finite labeled graph $\mathcal{G}$ is said to satisfy condition (I) if for every vertex v_i there exists distinct paths with distinct labeled edges both of whose sourses and terminals are the vertex v_i. We denote by $\mathcal{O}_{\mathcal{G},h_1,\dots,h_N}$ the C^*-symbolic crossed product $\mathcal{A}_{\mathcal{G},X} \rtimes_{\rho^{\mathcal{G},X}} \Lambda_{\mathcal{G}}$ for the C^*-symbolic dynamical system $(\mathcal{A}_{\mathcal{G},X}, \rho^{\mathcal{G},X}, \Sigma)$. Assume that there exists a faithful h_i-invariant probability measure on X.

Theorem 7.4. *Suppose that the labeled graph satisfies condition (I). $(h_1,\dots,h_N)$ is $\mathcal{G}$-minimal if and only if the C^*-algebra $\mathcal{O}_{\mathcal{G},h_1,\dots,h_N}$ is simple.*

Proof. Suppose that there exists a proper ideal $\mathcal{I}$ of $\mathcal{O}_{\mathcal{G},h_1,\dots,h_N}$. Since the labeled graph $\mathcal{G}$ satisfies condition (I), the C^*-symbolic dynamical system $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ satisfies condition (I) ([Ma2;Section 4]), so that $(\mathcal{A}_{\mathcal{G},X}, \rho^{\mathcal{G},X}, \Sigma)$ satisfies condition (I) by Theorem 6.5. Hence $J := \mathcal{I} \cap \mathcal{A}_{\mathcal{G},X}$ is a nonzero $\rho^{\mathcal{G},X}$-invariant saturated ideal of $\mathcal{A}_{\mathcal{G},X}$. If $J = \mathcal{A}_{\mathcal{G},X}$, then $\mathcal{A}_{\mathcal{G},X} \subset \mathcal{I}$ and $S_\alpha^* S_\alpha \in \mathcal{I}$ so that $S_\alpha \in \mathcal{I}$. Hence $\mathcal{I} = \mathcal{O}_{\mathcal{G},h_1,\dots,h_N}$. Therefore J is not a proper ideal of $\mathcal{A}_{\mathcal{G},X}$, and by Lemma 7.3 $(h_1,\dots,h_N)$ is not $\mathcal{G}$-minimal.

Next suppose that $(h_1, \dots, h_N)$ is not $\mathcal{G}$-minimal. By Lemma 7.3, there exists a proper $\rho^{\mathcal{G},X}$-invariant saturated ideal J of $\mathcal{A}_{\mathcal{G},X}$. The ideal $\mathcal{I}_J$ of $\mathcal{O}_{\mathcal{G},h_1,\dots,h_N}$ generated by J satisfies $\mathcal{I}_J \cap \mathcal{A}_{\mathcal{G},X} = J$ by Proposition 4.5. Hence $\mathcal{I}_J$ is a proper ideal of $\mathcal{O}_{\mathcal{G},\gamma_1,\dots,\gamma_N}$. $\square$

In [KW;Corollary 33], Kajiwara-Watatani have proved a similar result for the C^*-algebras from circle bimodules.

For a vertex $u \in V$ put $H_n[u] = H_n(u,u)$. Then we have

Proposition 7.5. *Suppose that $\mathcal{G}$ satisfies condition (I) and is irreducible. If there exists a path $(f_1, \dots, f_n) \in H_n[v_i]$ for some vertex $v_i \in V$ and $n \in \mathbb{N}$ such that the homeomorphism $\lambda(f_n)\circ\cdots\circ\lambda(f_1)$ on X_i is minimal, then $(h_1, \dots, h_N)$ is $\mathcal{G}$-minimal.*

Proof. Put $\xi = (f_1, \dots, f_n)$. Then $\lambda(\xi)$ is a minimal homeomorphism on X_i. For vertices $v_j, v_k \in V$, we may take paths $\gamma \in \cup_{m=1}^\infty H_m(v_i, v_j)$ and $\gamma' \in \cup_{m=1}^\infty H_m(v_k, v_i)$. Since for any $x \in X_i$, the orbit $\cup_{l=0}^\infty \lambda(\xi)^l(x)$ is dense in X_i, the set for any $y \in X_k$ $\cup_{l=0}^\infty \lambda(\gamma) \circ \lambda(\xi)^l \circ \lambda(\gamma')(y)$ is dense in X_j. Thus $(h_1, \dots, h_N)$ is $\mathcal{G}$-minimal. $\square$

The above discussions may be generalized to a λ-graph system with a family $\{h_1, \dots, h_N\}$ of homeomorphisms of a compact Hausdorff space X.

8. Irrational rotaton Cuntz-Kriger algebras

Let X be the circle $\mathbb{T}$ in the complex plane. Take an arbitrary finite family of real numbers $\{\theta_1, \dots, \theta_N\}$ with $\theta_i \in [0,1)$. Let $\alpha_i \in \mathrm{Aut}(C(\mathbb{T}))$ be the automorphisms of $C(\mathbb{T})$ defined by $\alpha_i(f)(t) = f(e^{2\pi\sqrt{-1}\theta_i}t), f \in C(\mathbb{T}), t \in \mathbb{T}$ for $i = 1, \dots, N$. Put $\Sigma = \{\alpha_1, \dots, \alpha_N\}$. Let $\mathcal{G}$ be a finite directed labeled graph (G, λ) over Σ with underlying finite directed graph $G = (V, E)$ and left resolving labeling $\lambda : E \to \Sigma$. We denote by $\{v_1, \dots, v_{N_0}\}$ the vertex set V. In [KW], Kajiwara-Watatani have studied the C^*-algebras constructed from circle correspondences. Their situation is more general than ours.

Assume that each vertex of V has both an incoming edge and an outgoing edge. Then we have a C^*-symbolic dynamical system as in the preceding sections, which we denote by $(\mathcal{A}_{\mathcal{G},\mathbb{T}}, \rho_{\theta_1,\dots,\theta_N}, \Sigma)$. Its C^*-symbolic crossed product is denoted by $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$. Let $A^{\mathcal{G}}$ be the matrix for $\mathcal{G}$ defined in (2.1).

Proposition 8.1. *The C^*-algebra $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ is the universal unital C^*-algebra generated by N partial isometries $S_i, i = 1, \dots, N$ and N_0 partial unitaries $U_j, j = 1, \dots, N_0$ subject to the following relations:*

$$\sum_{m=1}^{N} S_m^* S_m = 1, \qquad \sum_{j=1}^{N_0} U_j^* U_j = 1, \qquad U_i^* U_i = U_i U_i^*$$

$$U_i S_n = \sum_{j=1}^{N_0} A^{\mathcal{G}}(i, \alpha_n, j) e^{2\pi\sqrt{-1}\theta_n} S_n U_j,$$

$$S_n S_n^* U_i = U_i S_n S_n^* \qquad \text{for } i = 1, \dots, N_0,\ n = 1, \dots, N$$

such that

$$K_i(\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}) = \mathbb{Z}^{N_0}/(1 - A_{\mathcal{G}})\mathbb{Z}^{N_0} \oplus \mathrm{Ker}(1 - A_{\mathcal{G}}) \quad i = 0, 1,$$

where $A_{\mathcal{G}}$ is the $N_0 \times N_0$ matrix defined by $A_{\mathcal{G}}(i,j) = \sum_{\alpha\in\Sigma} A^{\mathcal{G}}(i, \alpha, j)$.

Proof. It suffices to show the formulae of K-groups. Since $K_i(\mathcal{A}_{\mathcal{G},\mathbb{T}}) = \mathbb{Z}^{N_0}, i = 0, 1$, by [Pim] (cf. [KPW]) the six term exact sequence of K-theory:

$$
\begin{array}{ccccc}
\mathbb{Z}^{N_0} & \xrightarrow{\mathrm{id}-A_{\mathcal{G}}} & \mathbb{Z}^{N_0} & \xrightarrow{\mathrm{id}} & K_0(\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}) \\
\uparrow & & & & \downarrow \\
K_1(\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}) & \underset{\mathrm{id}}{\longleftarrow} & \mathbb{Z}^{N_0} & \underset{\mathrm{id}-A_{\mathcal{G}}}{\longleftarrow} & \mathbb{Z}^{N_0}.
\end{array}
$$

holds so that one has the short exact sequences for $i = 0, 1$

$$
0 \longrightarrow \mathbb{Z}^{N_0}/(1 - A_{\mathcal{G}})\mathbb{Z}^{N_0} \longrightarrow K_i(\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}) \longrightarrow \mathrm{Ker}(1 - A_{\mathcal{G}}) \longrightarrow 0.
$$

They split because $\mathrm{Ker}(1 - A_{\mathcal{G}})$ is free so that the desired formulae hold. $\square$

We denote by $\mathcal{O}_{\mathcal{G}}$ the C^*-algebra of the labeled graph $\mathcal{G}$. It is isomorphic to a Cuntz-Krieger algebra (cf, [BP],[Ca],[Ma2],[Tom]). For $i, j = 1, \dots, N_0$, let $f_1, \dots, f_m$ be the set of edges in $\mathcal{G}$ whose source is v_i and terminal is v_j. Then we set $A^{\mathcal{G}_\theta}(i, j) = e^{2\pi\sqrt{-1}\theta_{k_1}} + \dots + e^{2\pi\sqrt{-1}\theta_{k_m}}$ formal sums for $\lambda(f_l) = \alpha_{k_l}, l = 1, \dots, m$. We have $N_0 \times N_0$ matrix $A^{\mathcal{G}_\theta}$ with entries in formal sums of nonnegative real numbers.

Proposition 8.2. *Suppose that the labeled graph $\mathcal{G}$ satisfies condition (I) and is irreducible. If there exists $n \in \mathbb{N}$ and $i = 1, \dots, N_0$ such that the (i, i)-component $(A^{\mathcal{G}_\theta})^n(i, i)$ of the n-th power of the matrix $A^{\mathcal{G}_\theta}$ contains an irrational angle of rotation, then $(\alpha_1, \dots, \alpha_N)$ is $\mathcal{G}$-minimal, so that the C^*-algebra $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ is simple, purely infinite.*

Proof. One knows that $(\alpha_1, \dots, \alpha_N)$ is $\mathcal{G}$-minimal by Proposition 7.5. It is easy to see that $(\mathcal{A}_{\mathcal{G}}, \rho_{\theta_1,\dots,\theta_N}, \Sigma)$ is effective. As the algebra $\mathcal{O}_{\mathcal{G}}$ is purely infinite, so is $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ by Theorem 5.5. $\square$

We will study the structure of both the algebra $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ and the fixed point algebra $\mathcal{F}_{\mathcal{G},\theta,\dots,\theta_N}$ of $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ under the gauge action. We denote by $\mathcal{F}_{\mathcal{G}}$ the fixed point algebra of $\mathcal{O}_{\mathcal{G}}$ under the gauge action.

Proposition 8.3. *Assume that the labeled graph $\mathcal{G}$ satisfies condition (I).*

(i) *$\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ is isomorphic to the crossed product $\mathcal{O}_{\mathcal{G}} \rtimes_{\gamma_{\theta_1,\dots,\theta_N}} \mathbb{Z}$ of the Cuntz-Krieger algebra $\mathcal{O}_{\mathcal{G}}$ of the labeled graph $\mathcal{G}$ by an automorphisms $\gamma_{\theta_1,\dots,\theta_N}$ of $\mathcal{O}_{\mathcal{G}}$.*

(ii) *$\mathcal{F}_{\mathcal{G},\theta,\dots,\theta_N}$ is an $A\mathbb{T}$-algebra, that is isomorphic to the crossed product $\mathcal{F}_{\mathcal{G}} \rtimes_{\gamma_{\theta_1,\dots,\theta_N}} \mathbb{Z}$ of the AF-algebra $\mathcal{F}_{\mathcal{G}}$ by the automorphism defined by the restriction of $\gamma_{\theta_1,\dots,\theta_N}$ to $\mathcal{F}_{\mathcal{G}}$.*

Proof. (i) Put $E_i = U_i^* U_i, i = 1, \dots, N_0$. The relations

$$
\sum_{j=1}^{N_0} E_j = 1, \qquad S_n^* E_i S_n = \sum_{j=1}^{N_0} A^{\mathcal{G}}(i, \alpha_n, j) E_j
$$

hold for $n = 1, \dots, N, i = 1, \dots, N_0$. Hence the C^*-subalgebra $C^*(S_n, E_i : n = 1, \dots, N, i = 1, \dots, N_0)$ of $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ generated by $S_n, E_i : n = 1, \dots, N, i =$

$1,\dots,N_0$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{\mathcal{G}}$ of the labeled graph $\mathcal{G}$. Put $U=\sum_{i=1}^{N_0}U_i$ a unitary. It is straightforwad to see the following relations hold:

$$US_nU^* = e^{2\pi\sqrt{-1}\theta_n}S_n, \qquad UE_i = E_iU = U_i,$$

for $n=1,\dots,N, i=1,\dots,N_0$. Since the algebra $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ is generated by S_n, E_i for $n=1,\dots,N, i=1,\dots,N_0$ and by putting

$$\gamma_{\theta_1,\dots,\theta_N}(S_n) = e^{2\pi\sqrt{-1}\theta_n}S_n, \qquad \gamma_{\theta_1,\dots,\theta_N}(E_i) = E_i$$

one sees that $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ is the crossed product of $C^*(S_n, E_i : n=1,\dots,N, i=1,\dots,N_0)$ by the automorphism $\gamma_{\theta_1,\dots,\theta_N}$.

(ii) The AF-algebra $\mathcal{F}_{\mathcal{G}}$ is regarded as the C^*-subalgebra of $\mathcal{O}_{\mathcal{G},\theta_1,\dots,\theta_N}$ generated by the elements of the form: $S_\mu E_i S_\nu^*, \mu,\nu\in\Lambda^*, |\mu|=|\nu|, i=1,\dots,N_0$. Under the identification, the algebra $\mathcal{F}_{\mathcal{G},\theta,\dots,\theta_N}$ is generated by $\mathcal{F}_{\mathcal{G}}$ and the above unitary U. By $\gamma_{\theta_1,\dots,\theta_N}(S_\mu E_i S_\nu^*) = e^{2\pi\sqrt{-1}(\theta_{\mu_1}+\cdots+\theta_{\mu_k}-\theta_{\nu_1}-\cdots-\theta_{\nu_k})}S_\mu E_i S_\nu^*$ for $\mu=(\mu_1,\dots,\mu_k),\ \nu=(\nu_1,\dots,\nu_k)\in\Lambda^k$, one knows that $\mathcal{F}_{\mathcal{G},\theta,\dots,\theta_N}$ is isomorphic to the crossed product $\mathcal{F}_{\mathcal{G}}\rtimes_{\gamma_{\theta_1,\dots,\theta_N}}\mathbb{Z}$ of $\mathcal{F}_{\mathcal{G}}$ by $\gamma_{\theta_1,\dots,\theta_N}$. By Proposition 6.8, one sees that $\mathcal{F}_{\mathcal{G},\theta,\dots,\theta_N}$ is an A$\mathbb{T}$-algebra. $\Box$

9. Irrational rotaton Cuntz algebras

In this section, we treat special cases of the previous section. We consider a labeled graph of N-loops with single vertex. Let $A=C(\mathbb{T})$ and $\Sigma=\{1,\dots,N\}, N>1$. Take real numbers $\theta_1,\dots,\theta_N\in[0,1)$. Define $\alpha_i(f)(z)=f(e^{2\pi\sqrt{-1}\theta_i}z)$ for $f\in C(\mathbb{T}), z\in\mathbb{T}$. We have a C^*-symbolic dynamical system $(C(\mathbb{T}),\alpha,\Sigma)$. Since $\alpha_i, i=1,\dots,N$ are automorphisms, the associated subshift is the full shift $\Sigma^{\mathbb{Z}}$. We denote by $\mathcal{O}_{\theta_1,\dots,\theta_N}$ the C^*-symbolic crossed product $C(\mathbb{T})\rtimes_\alpha\Sigma^{\mathbb{Z}}$. As the algebra $\mathcal{O}_{\theta_1,\dots,\theta_N}$ is the universal C^*-algebra generated by N isometries $S_i, i=1,\dots,N$ and one unitary U subject to the relations:

$$\sum_{j=1} S_jS_j^* = 1, \qquad S_i^*S_i = 1, \qquad US_i = e^{2\pi\sqrt{-1}\theta_i}S_iU, \qquad i=1,\dots,N,$$

it is realized as the ordinary crossd product $\mathcal{O}_N\rtimes_{\gamma_{\theta_1,\dots,\theta_N}}\mathbb{Z}$ of the Cuntz algebra $\mathcal{O}_N$ by the automorphism $\gamma_{\theta_1,\dots,\theta_N}$ defined by $\gamma_{\theta_1,\dots,\theta_N}(S_i)=e^{2\pi\sqrt{-1}\theta_i}S_i$. The K-groups are

$$K_0(\mathcal{O}_{\theta_1,\dots,\theta_N}) \cong K_1(\mathcal{O}_{\theta_1,\dots,\theta_N}) \cong \mathbb{Z}/(N-1)\mathbb{Z}.$$

By Theorem 5.5 and Theorem 7.4, one sees

Proposition 9.1. *The C^*-algebra $\mathcal{O}_{\theta_1,\dots,\theta_N}$ is simple if and only if at least one of $\theta_1,\dots,\theta_N$ is irrational. In this case, $\mathcal{O}_{\theta_1,\dots,\theta_N}$ is pure infinite.*

Remark. The algebra $\mathcal{O}_{\theta_1,\dots,\theta_N}$ is the crossed product $\mathcal{O}_N\rtimes_{\gamma_{\theta_1,\dots,\theta_N}}\mathbb{Z}$ of the Cuntz algebra $\mathcal{O}_N$ by the automorphism $\gamma_{\theta_1,\dots,\theta_N}$. The condition that at least one of $\theta_1,\dots,\theta_N$ is irrational is equivalent to the condition that the automorphisms $(\gamma_{\theta_1,\dots,\theta_N})^n$ are outer for all $n\in\mathbb{Z}, n\neq 0$. Hence by [Ki], the assertion for the simplicity of $\mathcal{O}_{\theta_1,\dots,\theta_N}$ in Proposition 9.1 holds.

We will study the fixed point algebra, denoted by $\mathcal{F}_{\theta_1,\dots,\theta_N}$, of $\mathcal{O}_{\theta_1,\dots,\theta_N}$ under the gauge action. It is generated by elements of the form $S_\mu f S_\nu^*$ for $f\in C(\mathbb{T}), |\mu|=|\nu|$.

Let $\mathcal{F}^k_{\theta_1,\dots,\theta_N}$ be the C^*-subalgebra of $\mathcal{F}_{\theta_1,\dots,\theta_N}$ generated by elements of the form $f \in C(\mathbb{T}), |\mu| = |\nu| = k$. The map

$$S_\mu f S_\nu^* \in \mathcal{F}^k_{\theta_1,\dots,\theta_N} \to f \otimes S_\mu S_\nu^* \in C(\mathbb{T}) \otimes M_{N^k}$$

yields an isomorphism between $\mathcal{F}^k_{\theta_1,\dots,\theta_N}$ and $C(\mathbb{T}) \otimes M_{N^k}$. Then the natural inclusion $\mathcal{F}^k_{\theta_1,\dots,\theta_N} \hookrightarrow \mathcal{F}^{k+1}_{\theta_1,\dots,\theta_N}$ through the identity $S_\mu f S_\nu^* = \sum_{i=1}^N S_{\mu i}\alpha_i(f) S_{\nu i}^*$ corresponds to the inclusion

$$
\begin{aligned}
& f \otimes e_{i,j} \in C(\mathbb{T}) \otimes M_{N^k} \\
\hookrightarrow & \begin{bmatrix} \alpha_1(f) \otimes e_{i,j} & & & 0 \\ & \alpha_2(f) \otimes e_{i,j} & & \\ & & \ddots & \\ 0 & & & \alpha_N(f) \otimes e_{i,j} \end{bmatrix} \in C(\mathbb{T}) \otimes M_{N^{k+1}}.
\end{aligned}
$$

For $\mu = (\mu_1, \dots, \mu_k) \in \Sigma^k$, we set $\alpha_\mu = \alpha_{\mu_k} \circ \cdots \circ \alpha_{\mu_1}$. Since $\mathcal{F}_{\theta_1,\dots,\theta_N}$ is an inductive limit of the inclusions $\mathcal{F}^k_{\theta_1,\dots,\theta_N} \hookrightarrow \mathcal{F}^{k+1}_{\theta_1,\dots,\theta_N}$, $k = 1, 2, \dots$ as in Proposition 6.8, it is an A$\mathbb{T}$-algebra.

Proposition 9.2. *The C^*-algebra $\mathcal{F}_{\theta_1,\dots,\theta_N}$ is simple if and only if $\theta_i - \theta_j$ is irrational for some $i, j = 1, \dots, N$.*

Proof. It is not difficult to prove the assertion directly by looking at the above inclusions $\mathcal{F}^k_{\theta_1,\dots,\theta_N} \hookrightarrow \mathcal{F}^{k+1}_{\theta_1,\dots,\theta_N}, k \in \mathbb{N}$. The following argument is a shorter proof by using [Ki]. Let $\mathcal{F}_N$ be the UHF-algebra of type N^∞, that is the fixed point algebra of $\mathcal{O}_N$ by the gauge action. By Proposition 8.3, $\mathcal{F}_{\theta_1,\dots,\theta_N}$ is the crossed product $\mathcal{F}_N \rtimes_{\gamma_{\theta_1,\dots,\theta_N}} \mathbb{Z}$ where $\gamma_{\theta_1,\dots,\theta_N}(S_\mu S_\nu^*) = e^{2\pi\sqrt{-1}(\theta_{\mu_1}+\cdots+\theta_{\mu_k}-\theta_{\nu_1}-\cdots-\theta_{\nu_k})} S_\mu S_\nu^*$ for $\mu = (\mu_1, \dots, \mu_k), \nu = (\nu_1, \dots, \nu_k) \in \Sigma^k$. Hence the automorphisms $\gamma_{\theta_1,\dots,\theta_N}$ is the product type automorphism $\prod^{\otimes} \mathrm{Ad}(u_\theta) = \mathrm{Ad}(u_\theta) \otimes \mathrm{Ad}(u_\theta) \otimes \cdots$ for the unitary $u_\theta = \begin{bmatrix} e^{2\pi\sqrt{-1}\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi\sqrt{-1}\theta_N} \end{bmatrix}$ in $M_N(\mathbb{C})$ under the canonical identification between $\mathcal{F}_N$ and $M_N \otimes M_N \otimes \cdots$. Then the condition that $\theta_i - \theta_j$ is irrational for some $i, j = 1, \dots, N$ is equivalent to the condition that $(\mathrm{Ad}(u_\theta))^n \neq \mathrm{id}$ for all $n \in \mathbb{Z}, n \neq 0$. In this case, the product type automorphisms $(\prod^{\otimes} \mathrm{Ad}(u_\theta))^n$ are outer for all $n \in \mathbb{Z}, n \neq 0$. Hence by [Ki], the assertion holds $\square$

For $\{\theta_1, \dots, \theta_N\}$ and $n \in \mathbb{N}$, put

$$S_n(\theta_1, \dots, \theta_N) = \{\theta_{i_1} + \cdots + \theta_{i_n} \mid i_1, \dots, i_n = 1, \dots, N\}.$$

then the sequence $\{S_n(\theta_1, \dots, \theta_N)\}_{n\in\mathbb{N}}$ of finite sets is said to be uniformly distributed in $\mathbb{T}$ ([Ki2]) if

$$\lim_{n\to\infty} \frac{1}{N^n} \sum_{i_1,\dots,i_n=1}^{N} f(e^{2\pi\sqrt{-1}(\theta_{i_1}+\cdots+\theta_{i_n})}) = \int_{\mathbb{T}} f(t)dt \qquad \text{for all } f \in C(\mathbb{T}).$$

The following lemma is easy

Lemma 9.3. *$\{S_n(\theta_1,\dots,\theta_N)\}_{n\in\mathbb{N}}$ is uniformly distributed in $\mathbb{T}$ if and only if $\theta_i-\theta_j$ is irrational for some $i,j=1,\dots,N$.*

Proof. $\{S_n(\theta_1,\dots,\theta_N)\}_{n\in\mathbb{N}}$ is uniformly distributed in $\mathbb{T}$ if and only if

$$\lim_{n\to\infty}\frac{1}{N^n}\sum_{i_1,\dots,i_n=1}^{N} e^{2\pi\sqrt{-1}\ell(\theta_{i_1}+\dots+\theta_{i_n})}=0 \qquad \text{for all } \ell\in\mathbb{Z},\ell\neq 0. \tag{9.1}$$

Since $\sum_{i_1,\dots,i_n=1}^{N} e^{2\pi\sqrt{-1}\ell(\theta_{i_1}+\dots+\theta_{i_n})}=(e^{2\pi\sqrt{-1}\ell\theta_1}+\dots+e^{2\pi\sqrt{-1}\ell\theta_N})^n$, the condition (9.1) holds if and only if

$$|e^{2\pi\sqrt{-1}\ell\theta_1}+\dots+e^{2\pi\sqrt{-1}\ell\theta_N}|<N \qquad \text{for all } \ell\in\mathbb{Z},\ell\neq 0. \tag{9.2}$$

The condition (9.2) is equivalent to the condition that $\theta_i-\theta_j$ is irrational for some $i,j=1,\dots,N$. $\square$

Thereofre we have

Theorem 9.4. *For $\theta_1,\dots,\theta_N\in[0,1)$, the following conditions are equivalent:*

(i) *$\theta_i-\theta_j$ is irrational for some $i,j=1,\dots,N$.*
(ii) *$\mathcal{F}_{\theta_1,\dots,\theta_N}$ is simple.*
(iii) *$\mathcal{F}_{\theta_1,\dots,\theta_N}$ has real rank zero.*

Proof. The equivalence between (i) and (ii) follows from Proposition 9.2. It suffices to show the equivalence between (i) and (iii). Since

$$\mathrm{Sp}(\underbrace{u_\theta\otimes\dots\otimes u_\theta}_{n})=S_n(\theta_1,\dots,\theta_N)$$

and $\gamma_{\theta_1,\dots,\theta_N}$ is a product type automorphism on $\prod^{\otimes} Ad(u_\theta)$ on the UHF-algebra $\mathcal{F}_N$, by [Ki2;Lemma 5.2] the crossed product $\mathcal{F}_N\rtimes_{\gamma_{\theta_1,\dots,\theta_N}}\mathbb{Z}$ has real rank zero if and only if $S_n(\theta_1,\dots,\theta_N)$ is uniformly distrubuted in $\mathbb{T}$. $\square$

We note that by [Ki;Lemma 5.2], the crossed product $\mathcal{F}_N\rtimes_{\gamma_{\theta_1,\dots,\theta_N}}\mathbb{Z}$ has real rank zero if and only if $\mathcal{F}_{\theta_1,\dots,\theta_N}$ has a unique trace.

Consequently we obtain

Theorem 9.5. *For $\theta_1,\dots,\theta_N\in[0,1)$, suppose that there exist $i,j=1,\dots,N$ such that $\theta_i-\theta_j$ is irrational. Then the C^*-algebra $\mathcal{F}_{\theta_1,\dots,\theta_N}$ is a unital simple $A\mathbb{T}$-algebra of real rank zero with a unique tracial state such that*

$$K_0(\mathcal{F}_{\theta_1,\dots,\theta_N})\cong\mathbb{Z}[\frac{1}{N}], \qquad K_1(\mathcal{F}_{\theta_1,\dots,\theta_N})\cong\mathbb{Z}.$$

Hence $\mathcal{F}_{\theta_1,\dots,\theta_N}$ is the Bunce-Deddens algebra of type N^∞.

Proof. Since $K_i(C(\mathbb{T}\otimes M_{N^k})=\mathbb{Z}, i=0,1$ and the homomorphisms in Proposition 6.8 yield the N-multiplications on $K_0(C(\mathbb{T}\otimes M_{N^k})=\mathbb{Z}\to K_0(C(\mathbb{T}\otimes M_{N^{k+1}})=\mathbb{Z}$ and the identities on $K_1(C(\mathbb{T}\otimes M_{N^k})=\mathbb{Z}\to K_1(C(\mathbb{T}\otimes M_{N^{k+1}})=\mathbb{Z}$, we get the K-theory formulae by Proposition 6.8. The obtained isomorphism from $K_0(\mathcal{F}_{\theta_1,\dots,\theta_N})$ to $\mathbb{Z}[\frac{1}{N}]$ preserves their order and maps the unit 1 of $\mathcal{F}_{\theta_1,\dots,\theta_N}$ to 1 in $\mathbb{Z}[\frac{1}{N}]$. Hence $\mathcal{F}_{\theta_1,\dots,\theta_N}$ is isomorphic to the Bunce-Deddens algebra of type N^∞. $\square$

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